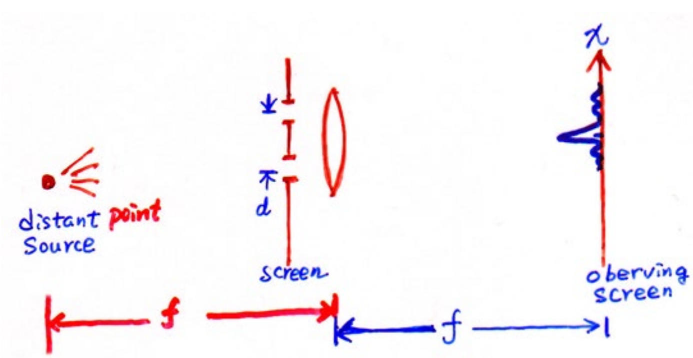
# Chapter 5  Modern Theory of Optical Coherence

## 5.1 Heuristic Introduction to Optical Coherence

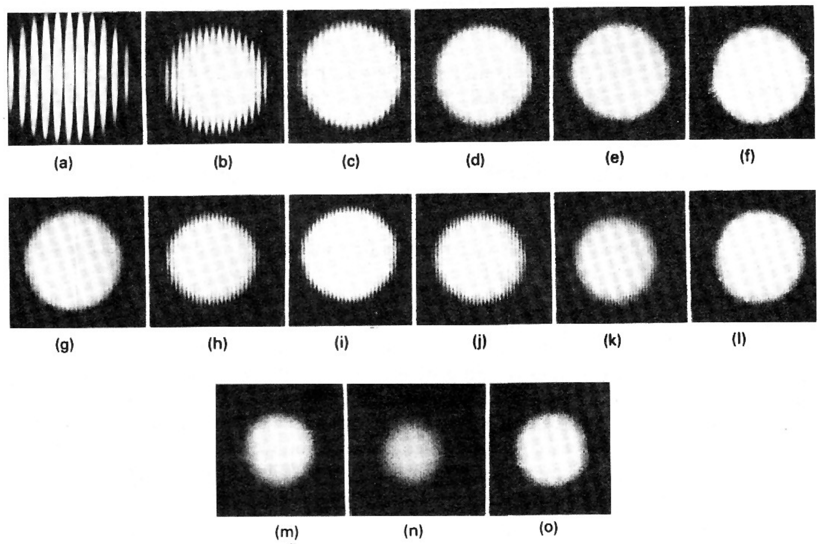
To understand the relationship of the performance of an interferometer with the optical source used, it is helpful to know the coherence property of light field. (see: A. Derode and M. Fink, *The notion of coherence in optics*, Euro. J. Phys. 15, 81-90 (1994)). This chapter will focus on some fundamental concepts of the coherence properties of light.

Let us first consider the intensity of an optical field, which is superposed from two fields denoted by $\{\Psi_i\}_{i=1}^{2}$ from an *incoherent light source*. We can present the resultant intensity as $I = \sum_i |\Psi_i|^2$. Now consider the case that a *coherent source* is used, the resultant intensity becomes the absolute square of the sum of the individual wave by $I = |\sum_{i=1,2} \Psi_i|^2$. What about the resultant intensity superposed from two fields from a *partially coherent light*? This is an important issue in viewing that there exist a wide variety of situations in which the partial coherence in the optical field needs to be considered. To answer the question, it is helpful by studying the following heuristic experiments, which had been performed by Young, et al. in 1800.

(1) **Experiment 1:** Young's double-slit interferometry with a **monochromatic point** light source.



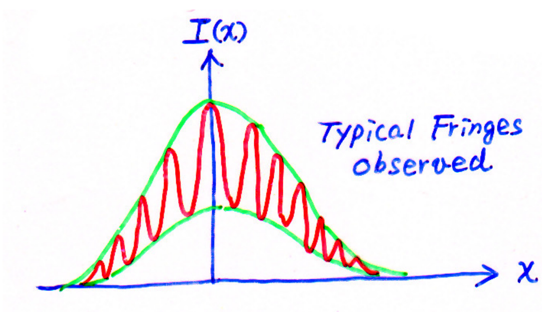
The typical fringe patterns formed on the observing screen were shown on the left.

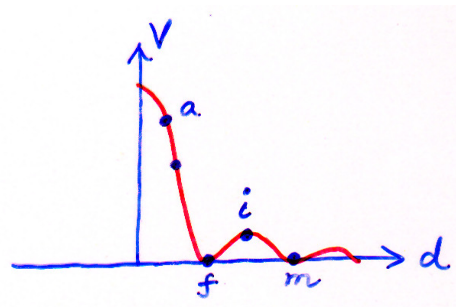



We can characterize these fringe patterns with the parameter of **Visibility**, which is defined by

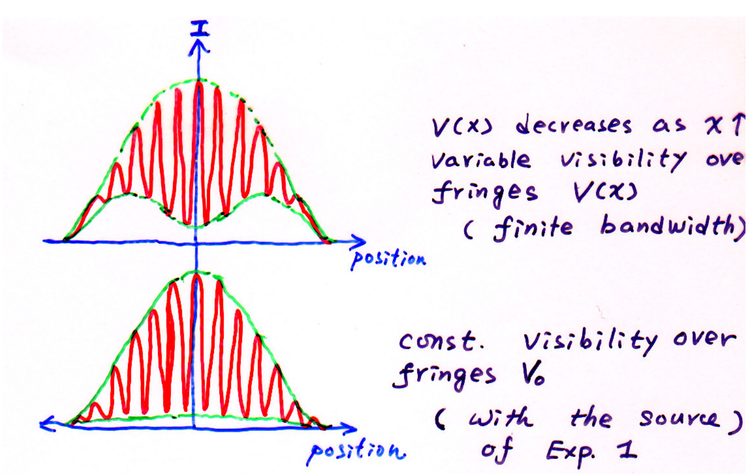
$$V \equiv \frac{(I_{\max} - I_{\min})}{(I_{\max} + I_{\min})}.$$

Quantitative analysis will show the visibility is constant for the fringe patterns as depicted below:




**(2) Experiment 2:** Young's double-slit interferometry with a **finite spectral bandwidth point** light source

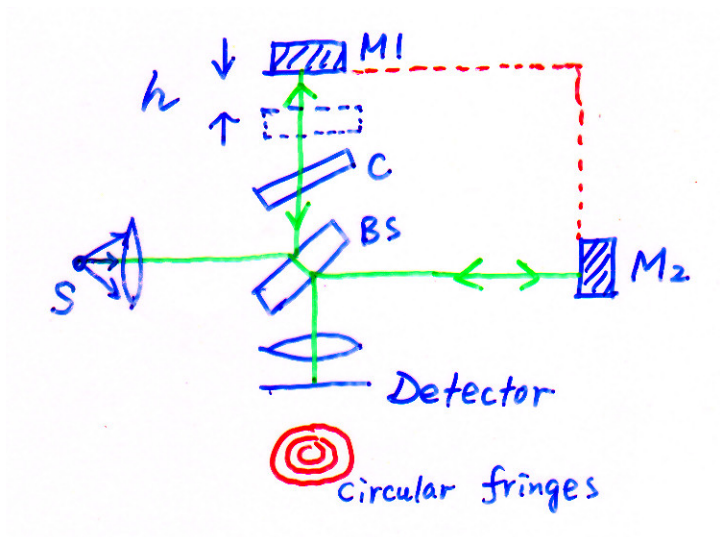



The experimental arrangement is the same as that of Experiment 1 but using a different light source with a finite spectral bandwidth (for example $\sim 50$ Å). In this case, the visibility can decrease as *x* increases (*variable* *visibility over fringe pattern*). This comparison can be summarized as: Source in Experiment 1 can retain visibility over much larger path difference; but source 2 with finite bandwidth rapidly loses visibility (coherence) as path difference increases.

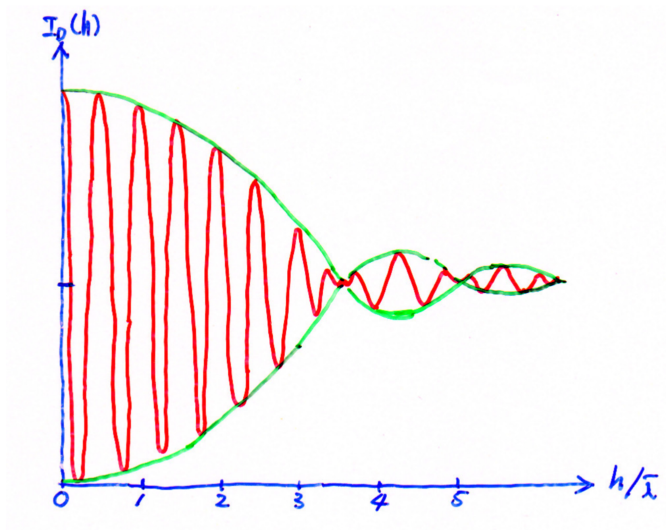
Now let us consider optical coherence revealed by a different type of interferometry.

**(3) Experiment 3:** Michelson Interferometry for Temporal Coherence

Consider an optical field disturbance at a position ***P*** and at the instant $t$ $u(P,t) = A(P,t)e^{j2\pi\bar{\nu}t}$ with a central frequency $\bar{\nu}$ and a finite bandwidth $\Delta\nu$. Here $A(P,t) \simeq A(P,t+\tau)$ if $\tau \ll 1/\Delta\nu = \tau_c$ = coherence time of the source.




If $M_1$ is moved from the position required for equal path lengths in the two arms of the interferometer, a relative time delay is induced between the two interfering beams. A mirror movement of $\bar{\lambda}/2 = c/(2\bar{\nu})$ corresponds to two neighboring bright fringes. Superimposed on this rapid oscillation of intensity is a gradually tapering envelope of fringe modulation, caused by the finite bandwidth of the source and the gradual de-correlation of the complex envelope of the light $A(P,t)$ as the path length difference increases.




## 5.2 Mathematical Description of the Experiment 3

The response of the detector is governed by the intensity of the optical wave falling on its surface $I_D(h) \equiv <|K_1\vec{u}(t)+K_2\vec{u}(t+\frac{2h}{c})|^2>_t$ , where $< \ldots >_t$ denotes a long time averaging. If $u(P, t)$ is an ergodically stochastic process, $<\ldots>_t$ can be replaced by the ensemble average $<\ldots>$.

$$\begin{aligned}I_D(h) &= K_1^2 <|\vec{u}(t)|^2>_t + K_2^2 <|\vec{u}(t+\frac{2h}{c})|^2>_t \\ &+ K_1K_2 < \vec{u}(t+\frac{2h}{c})\vec{u}^*(t)>_t + K_1K_2 < \vec{u}^*(t+\frac{2h}{c})\vec{u}(t)>_t\end{aligned}.$$

$$\begin{aligned}I_D(h) &= (K_1^2+K_2^2)I_o + K_1K_2\Gamma(\frac{2h}{c}) + K_1K_2\Gamma^*(\frac{2h}{c}) \\ &= (K_1^2+K_2^2)I_o + 2K_1K_2\,\mathrm{Re}\{\Gamma(\frac{2h}{c})\}\end{aligned}.$$

Here $\Gamma(\tau) = < \vec{u}(t+\tau)\vec{u}^*(t) >_t =$ autocorrelation (*i.e.*, the self coherence) function of the analytic signal $u(t)$. We can define a normalized version of the self coherence function by

$\tilde{\gamma}(\tau) = \Gamma(\tau)/\Gamma(0) = \gamma(\tau)e^{-j[2\pi\bar{\nu}\tau - \alpha(\tau)]} = |\tilde{\gamma}(\tau)|\, e^{j\arg[\tilde{\gamma}(\tau)]} =$ *complex degree of coherence* of the light. Note $\bar{\nu}\,\tau = \frac{c}{\bar{\lambda}} \cdot \frac{2h}{c} = \frac{2h}{\bar{\lambda}}$. The fringe pattern detected by the Michelson interferometer can then be expressed as

$I_D(h) = 2K^2 I_o\{1 + \gamma(\frac{2h}{c})\cos[2\pi(\frac{2h}{\bar{\lambda}}) - \alpha(\frac{2h}{c})]\}$ =fringe pattern (or *interferogram*).

■ When $h \to 0$ $\quad \gamma(\frac{2h}{c}) \sim \gamma(0) = 1$ and $\alpha(\frac{2h}{c}) = \arg[\tilde{\gamma}(\frac{2h}{c})] + 2\pi\bar{\nu}(\frac{2h}{c}) \sim 0$

$$\therefore \quad I_D(h) \xrightarrow{h \to 0} 2K^2 I_o\{1 + \cos[2\pi(\frac{2h}{\bar{\lambda}})]\}.$$

■ When $\frac{2h}{c} > \tau_c = \frac{1}{\Delta\nu}, \quad \gamma(2h/c) \sim 0$

$$\therefore \quad I_D(h) \sim 2K^2 I_o.$$

Visibility of the fringes becomes

$$V(h) \equiv \frac{(I_{\max} - I_{\min})}{(I_{\max} + I_{\min})} = |\tilde{\gamma}(2h/c)| = \gamma(2h/c) \to \begin{cases} 0, & as\ 2h/c > \tau_c \\ 1, & as\ 2h/c \to 0 \end{cases}.$$

The *temporal coherence* relates to the ability of two relative delayed light beams to form fringes.

### 5.2.2 Interferogram and Power Spectral Density of a Light Beam

Assuming $u$ to be an **analytic** signal with

$$u(t)=u^{(r)}(t)+ju^{(i)}(t)=u^{(r)}(t)-\frac{j}{\pi}\int_{-\infty}^{\infty}\frac{u^{(r)}(\xi)}{t-\xi}d\xi$$
$$=u^{(r)}(t)-\frac{j}{\pi}\left[\frac{1}{t}\otimes u^{(r)}(t)\right]=\left[\delta(t)-\frac{j}{\pi t}\right]\otimes u^{(r)}(t).$$

By use of the relationship between the ensemble averaging and the autocorrelation function $\Gamma(t+\tau,t)=E[u(t+\tau)u(t)]=\int u(t+\tau)u(t)\cdot p_U[u(t+\tau),u(t)]\cdot du(t+\tau)du(t)$,

we can define a power spectral density of analytic signal $u$ as

$$\begin{aligned}\mathcal{P}_U(\nu)&=\lim_{T\to\infty}\frac{E[|U_T(\nu)|^2]}{T}=\int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}rect(\frac{t+\tau}{T})rect(\frac{t}{T})\Gamma(t+\tau,t)dt\cdot e^{j2\pi\nu\tau}d\tau\\&=\int_{-\infty}^{\infty}\Gamma_U(t+\tau,t)e^{j2\pi\nu\tau}d\tau=FT[\Gamma_U(\tau)]\\&=FT\{\Gamma_U{}^{(r,r)}(\tau)+\Gamma_U{}^{(i,i)}(\tau)+j\,[\Gamma_U{}^{(i,r)}(\tau)-\Gamma_U{}^{(r,i)}(\tau)]\}\\&=2FT\{\Gamma_U{}^{(r,r)}(\tau)+j\Gamma_U{}^{(i,r)}(\tau)\}=2\mathcal{P}_U{}^{(r,r)}(\nu)+2\,\mathrm{sgn}(\nu)\mathcal{P}_U{}^{(r,r)}(\nu)\\&=\begin{cases}4\mathcal{P}_U{}^{(r,r)}(\nu); & \nu>0\\ 0; & \nu<0\end{cases}\end{aligned},$$

where $\mathcal{P}_U{}^{(r,r)}(\nu)$ is the *power spectral density* of the real-valued optical disturbance $u^{(r)}(t)$.

$\therefore\quad \gamma(\tau)=\int_0^{\infty}4\mathcal{P}^{(r,r)}(\nu)e^{-j2\pi\nu\tau}d\nu\Big/\int_0^{\infty}4\mathcal{P}^{(r,r)}(\nu)d\nu=\int_0^{\infty}\hat{\mathcal{P}}^{(r,r)}(\nu)e^{-j2\pi\nu\tau}d\nu$, where $\hat{\mathcal{P}}^{(r,r)}(\nu)$ denotes *the normalized power spectral density*.

Consider the *power spectral density* of some typical light sources:

■ Low-pressure gas discharge lamp with an inhomogeneous broadening due to Doppler effect

$$\hat{\mathcal{P}}(\nu) \simeq \frac{2\sqrt{\ln 2}}{\sqrt{\pi}\Delta\nu} e^{-(2\sqrt{\ln 2}\,\frac{\nu-\bar{\nu}}{\Delta\nu})^2}$$

$$\Rightarrow \quad \gamma(\tau) = e^{-(\frac{\pi}{2}\frac{\Delta\nu\,\tau}{\sqrt{\ln 2}})^2} \cdot e^{-j2\pi\bar{\nu}\tau} \quad and \quad \alpha(\tau) = \arg[\gamma(\tau)] = 0 .$$

■ High-pressure gas discharge lamp with a Lorentzian lineshape from strong collision

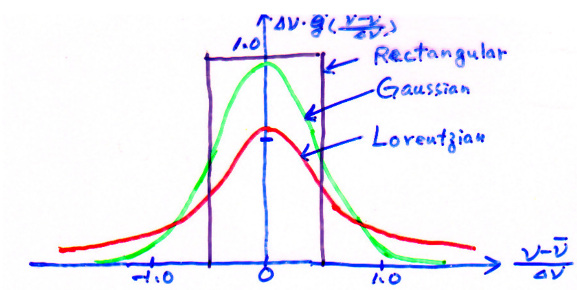
$$\hat{\mathcal{P}}(\nu) \simeq \frac{2}{\pi\Delta\nu} \cdot \frac{1}{1+(2\,\frac{\nu-\bar{\nu}}{\Delta\nu})^2} .$$

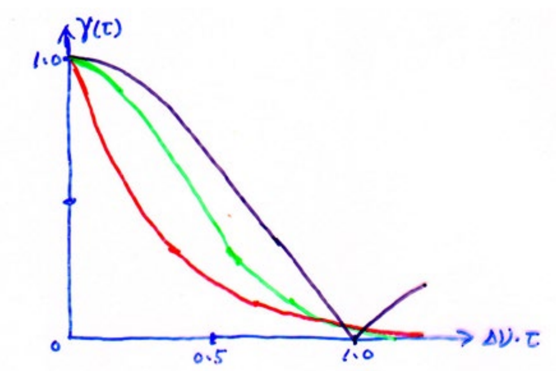
$$\Rightarrow \quad \gamma(\tau) = e^{-\pi\Delta\nu\cdot|\tau|} \cdot e^{-j2\pi\bar{\nu}\tau}$$

■ Source with rectangular lineshape

$$\hat{\mathcal{P}}(\nu) \simeq \frac{1}{\Delta\nu} \cdot rect(\frac{\nu-\bar{\nu}}{\Delta\nu})$$

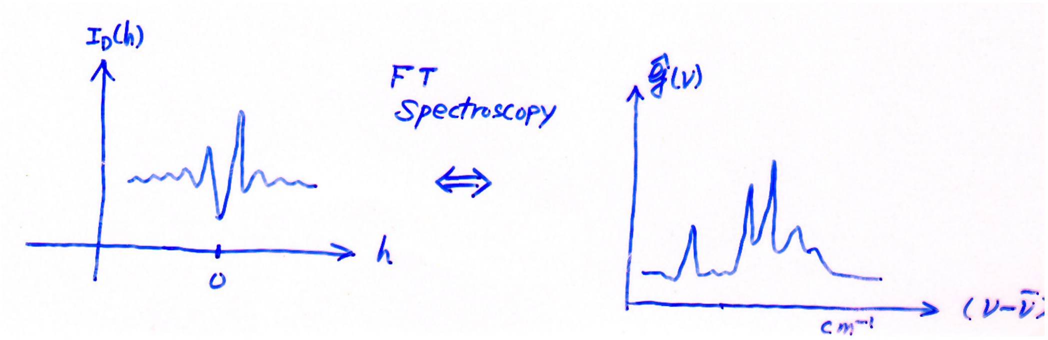
$$\Rightarrow \quad \gamma(\tau) = Sinc(\Delta\nu\cdot\tau) \cdot e^{-j2\pi\bar{\nu}\tau}$$




If $\hat{\mathcal{P}}(\nu)$ is an even function of $(\nu-\bar{\nu})$, $\Rightarrow \quad \gamma(\tau) = real - value \cdot e^{-j2\pi\bar{\nu}\tau}$, and

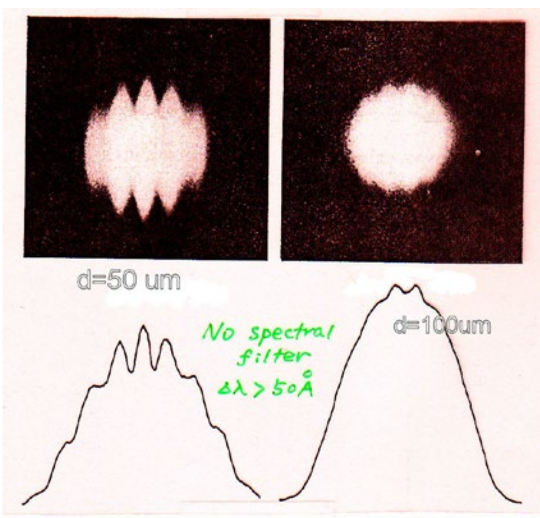
$$\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 \, d\tau = \begin{cases} \dfrac{0.664}{\Delta\nu}; & \text{Gaussian} \\ \dfrac{0.318}{\Delta\nu}; & \text{Lorentzian} \\ \dfrac{1}{\Delta\nu}; & \text{Rectangular} \end{cases} .$$




## 5.3 Spatial Coherence

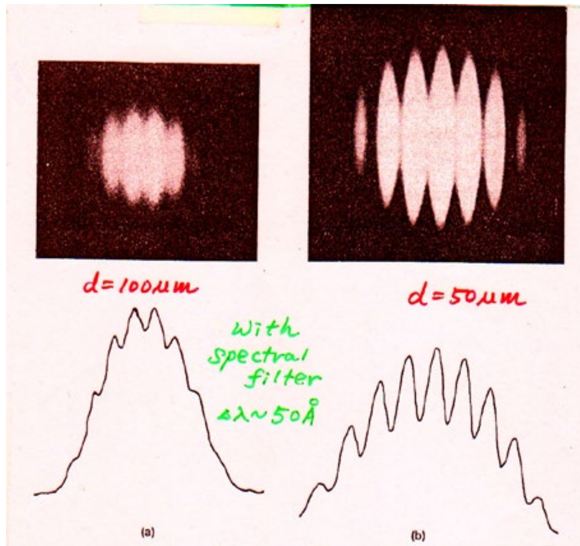
What are the influences of a light source with an extended emitting area on a Young's double-slit experiment. To analyze the influences, we first review the effects of

■ *Pinhole Separation* and *Spectral Bandwidth* of the Light Source used



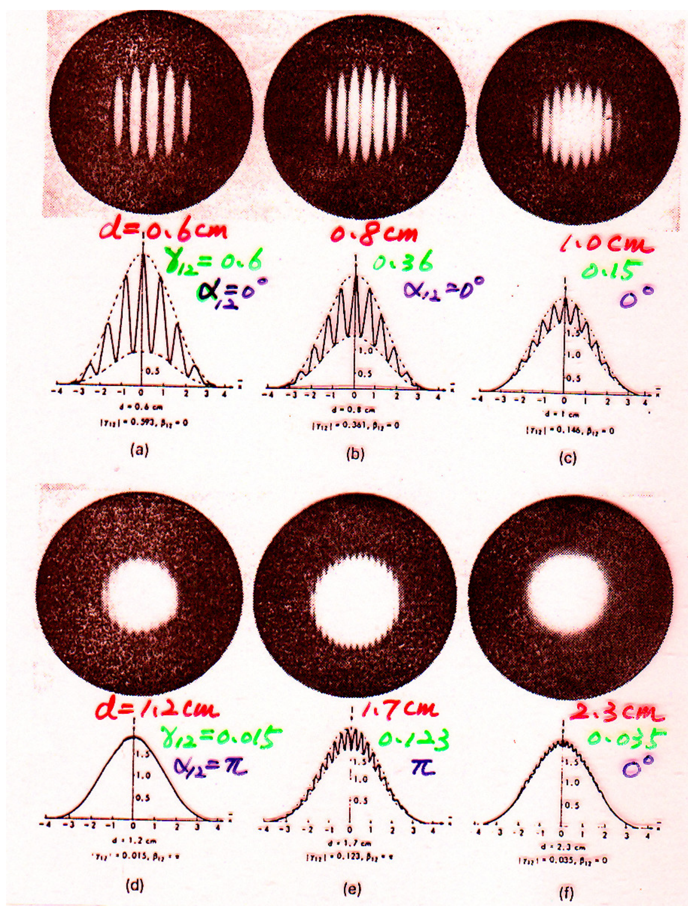

For a broadband light source (without using a spectral filter), the resultant interferograms with two pinhole separations (d=50 μm and 100μm) are shown on the left.

If we insert a spectral filter behind the light source to reduce the bandwidth to *5 nm*, then the following different patterns will be observed

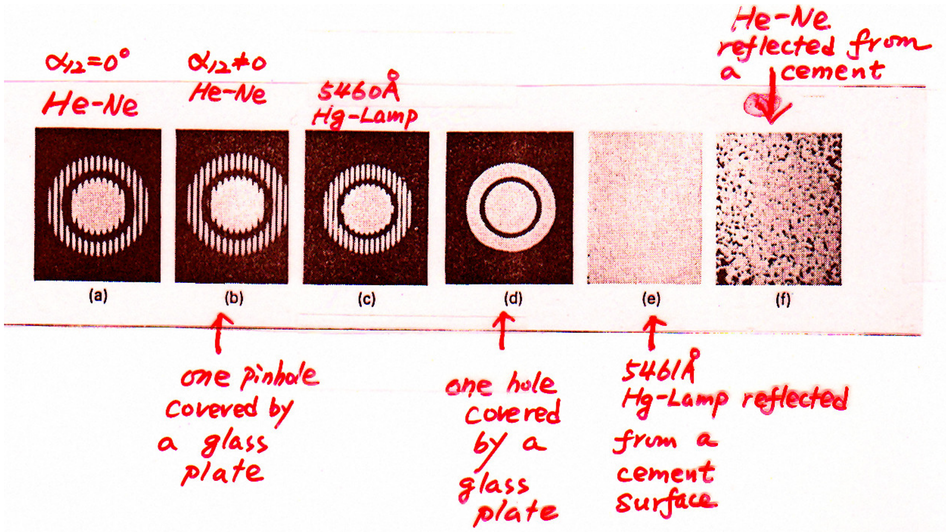



■ Two-Beam Interference Experiment with *Partially Coherent Light*

Now let us reduce the light coherence ($\gamma_{12}$) between the fields sampled by the two pinholes, which can be changed by increasing the separation of the pinholes used.



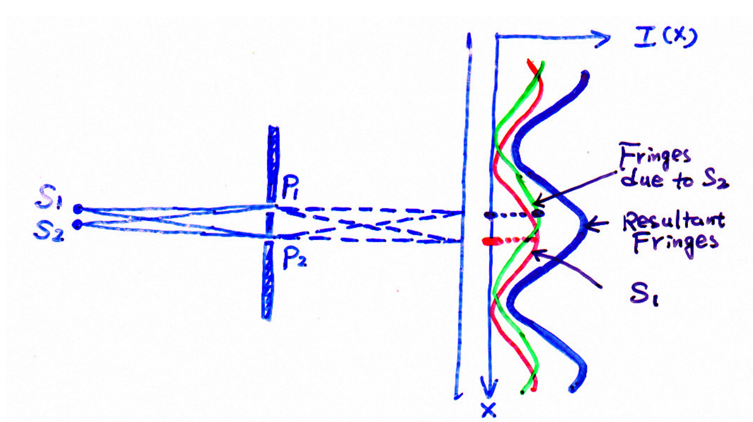
■ Two-Beam Interference Experiment with Two Different Light Sources




Now an intuitive picture of Young's two-beam interference experiment may be drawn below:

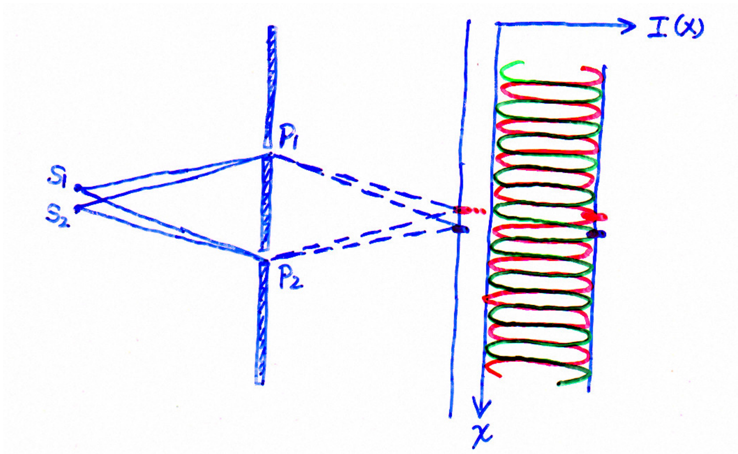
1. If the light is approximately monochromatic and originates from a single point source ($S_1$), sinusoidal fringes of high contrast are observed.
2. If a second point source ($S_2$) of the same wavelength as the first but radiating independently is added, a second fringe pattern is generated.

The period of these two patterns are the same but the positions of zero path length difference are shifted relatively.

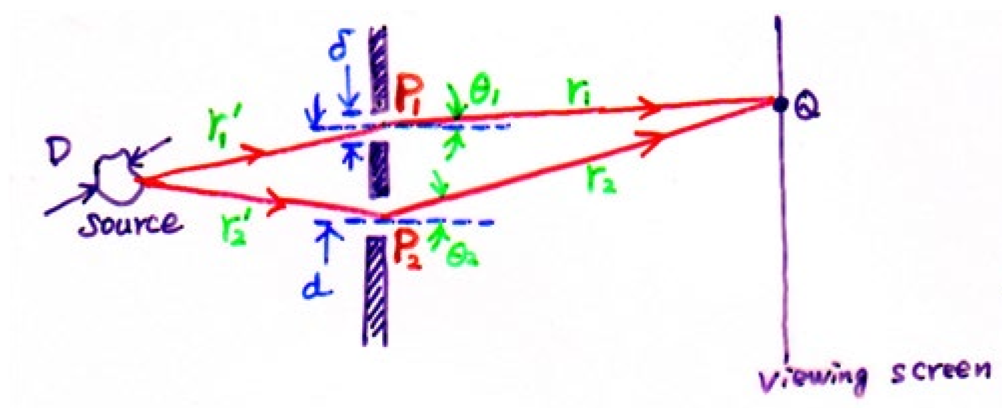



If the pinhole separation is small, the fringes are very coarse, the shift of one fringes with respect to the other is a negligible fraction of a period as illustrating on the left.

If the pinhole separation is large, then the fringe period is small, and the fringe is shifted by a significant fraction of its period.




### 5.3.2 Mathematical Description of Spatial Coherence




Let us consider the detector response at the position $Q$ and at the instant $t$ as

$I(Q) = \langle u^*(Q,t)u(Q,t) \rangle$. By using *Huygens-Fresnel principle*, the field at $Q$ in a Young's interferometer can be described by

$$u(Q,t) = K_1 u(P_1, t - \frac{r_1}{c}) + K_2 u(P_2, t - \frac{r_2}{c}), \text{ where}$$

$$K_1 \simeq \int_{\Sigma_1} \frac{\chi(\theta_1)}{j\bar{\lambda}\, r_1'} dS_1 \,; \quad K_2 \simeq \int_{\Sigma_2} \frac{\chi(\theta_2)}{j\bar{\lambda}\, r_2'} dS_2 \,.$$

Define

$$I_{P_1}(Q) = I_1(Q) = |K_1|^2 \langle |u(P_1, t - \frac{r_1}{c})|^2 \rangle$$

$$I_{P_2}(Q) = I_2(Q) = |K_2|^2 \langle |u(P_2, t - \frac{r_2}{c})|^2 \rangle$$

and $\Gamma_{12}(\tau) \equiv \langle u(P_1, t+\tau) u^*(P_2, t) \rangle$ where $\tau = |r_2 - r_1|/c$.

$$\therefore \quad I(Q) = I_1(Q) + I_2(Q) + 2K_1K_2 \,\mathrm{Re}\left[\Gamma_{12}(\frac{|r_2 - r_1|}{c})\right].$$

By invoking the Schwarz's inequality, we now obtain

$$|\Gamma_{12}(\tau)| \le \sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}\,,$$

where $\Gamma_{11}(0)$ and $\Gamma_{22}(0)$ are the *self-coherence function* of the light at pinhole $P_1$ and $P_2$, respectively. We can define a *complex degree of coherence* by normalizing the cross-correlation function

$$\tilde{\gamma}_{12}(\tau) \equiv \frac{\Gamma_{12}(\tau)}{[\Gamma_{11}(0)\Gamma_{22}(0)]^{1/2}} = \text{complex degree of coherence.}$$

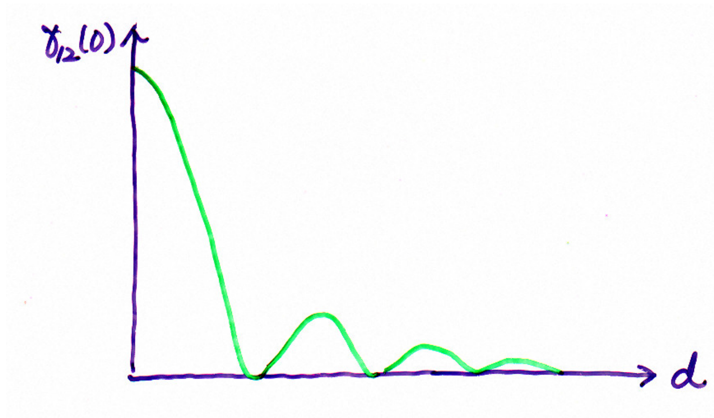
Let $\tilde{\gamma}_{12}(\tau) = \gamma_{12}(\tau)\, e^{-j[2\pi\bar{\nu}\tau - \alpha_{12}(\tau)]}$, then

$$I(Q) = I_1(Q) + I_2(Q) + 2\sqrt{I_1(Q)I_2(Q)} \cdot \gamma_{12}(\frac{r_2 - r_1}{c}) \cdot \cos[2\pi\bar{\nu}(\frac{r_2 - r_1}{c}) - \alpha_{12}(\frac{r_2 - r_1}{c})]\,,$$

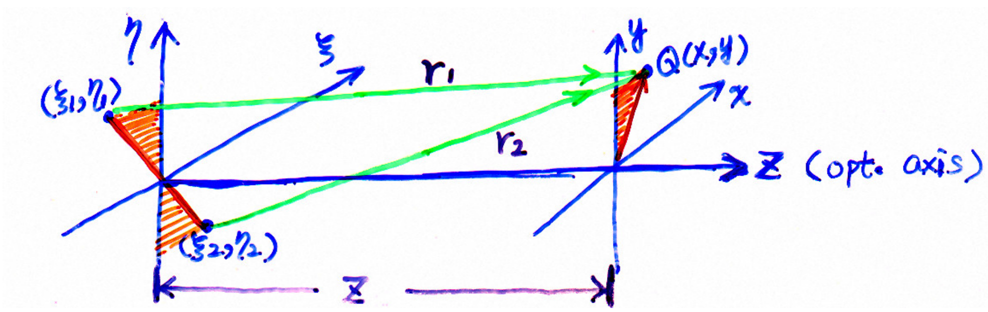
and when $Q$ lies on the symmetric axis of the pinhole $P_1$ and $P_2$, $r_2 - r_1 = 0$, which yields

$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\gamma_{12}(0) = \gamma_{12}(0) \quad when \quad I_1 = I_2.$$

A description of how $\gamma_{12}(0)$ changes with the distance between $P_1$ and $P_2$ is a description of *the spatial coherence* of the light striking the pinhole plane.




### 5.3.3 Geometric Considerations of Spatial Coherence




For simplicity, let us use paraxial approximation,

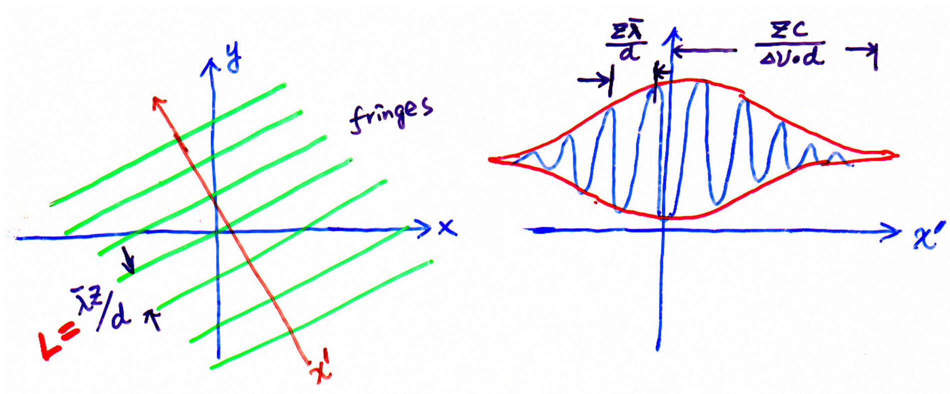
$$r_1 = [z^2 + (\xi_1 - x)^2 + (\eta_1 - y)^2]^{1/2} \simeq z + \frac{(\xi_1 - x)^2}{2z} + \frac{(\eta_1 - y)^2}{2z}$$

$$r_2 = [z^2 + (\xi_2 - x)^2 + (\eta_2 - y)^2]^{1/2} \simeq z + \frac{(\xi_2 - x)^2}{2z} + \frac{(\eta_2 - y)^2}{2z} \cdot$$

$$\therefore r_2 - r_1 \simeq \frac{(\xi_2 - x)^2 + (\eta_2 - y)^2 - (\xi_1 - x)^2 - (\eta_1 - y)^2}{2z} = \frac{1}{2z}[\rho_2{}^2 + \rho_1{}^2 - 2\Delta\xi \cdot x - 2\Delta\eta \cdot y],$$

where $\rho_i{}^2 = \xi_i{}^2 + \eta_i{}^2, \quad \Delta\xi = \xi_2 - \xi_1, \quad \Delta\eta = \eta_2 - \eta_1$.

Let $\bar{\lambda} = c/\bar{\nu}, \quad L = \bar{\lambda} z/d, \quad d = \sqrt{(\Delta\xi)^2 + (\Delta\eta)^2}$ = pinhole separation




$$\therefore \quad N = total\ \ fringes\ observed = (\frac{2zc}{\Delta\nu \cdot d}) \Big/ (\frac{zc}{d\,\bar{\nu}}) = 2 \cdot \frac{\bar{\nu}}{\Delta\nu} = 2 \cdot \frac{central\ \ freqeuncy\ of\ \ light}{spectral\ bandwidth}.$$

### 5.3.4 Interference under Quasi-Monochromatic Conditions

Considering a narrowband light source $\Delta\nu \ll \bar{\nu}$ and $[(r_2 + r_2') - (r_1 + r_1')]/c \ll \tau_c$ (the total time delay difference is much smaller than the coherent time $\tau_c = 1/\Delta\nu$ ), which are called *quasi-monochromatic conditions*, the complex coherence function $\Gamma_{12}(\tau) \simeq < u(P_1,t)u^*(P_2,t) > e^{-j2\pi\bar{\nu}\tau} = J_{12}\, e^{-j2\pi\bar{\nu}\tau}$. Here $J_{12}$ is the *mutual intensity* at $P_1$ and $P_2$. The corresponding complex degree of coherence becomes

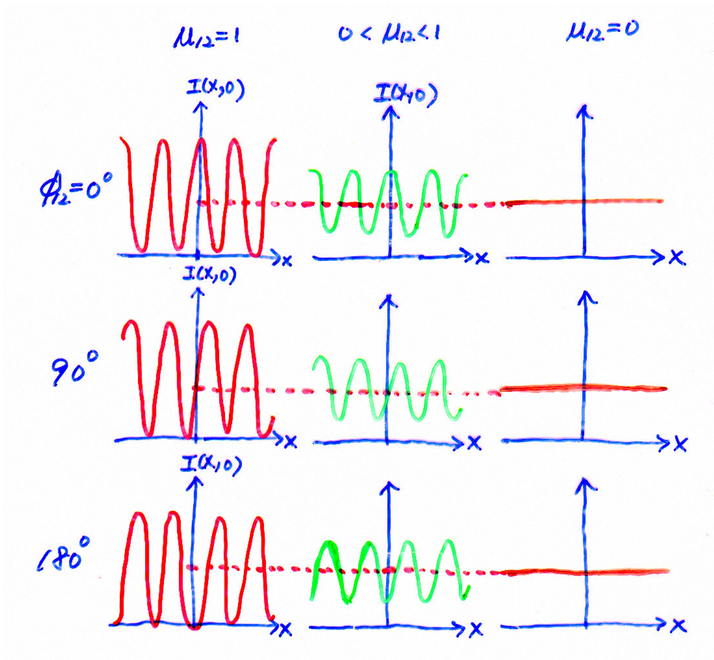
$\gamma_{12}(\tau) = \dfrac{\Gamma_{12}(\tau)}{\Gamma_{12}(0)} \simeq \mu_{12}\, e^{-j2\pi\bar{\nu}\tau}$ with $\mu_{12}$ denoting the *complex coherence factor*.

Therefore, the fringe pattern can be expressed as

$$\begin{aligned} I(Q) = I(x,y) &= I_1 + I_2 + 2K_1K_2J_{12}\cos[\frac{2\pi}{\bar{\lambda} z_2}(\Delta\xi \cdot x + \Delta\eta \cdot y) + \phi_{12}] \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mu_{12}\cos[\frac{2\pi}{\bar{\lambda} z_2}(\Delta\xi \cdot x + \Delta\eta \cdot y) + \phi_{12}] \end{aligned},$$

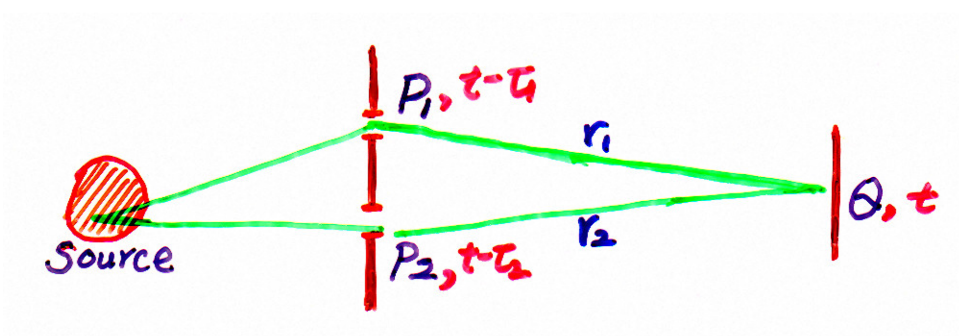
where

$\phi_{12} = \arg[J_{12}] - \frac{\pi}{\bar{\lambda} z_2}(\rho_2^{\,2} + \rho_1^{\,2}) = \alpha_{12}(0) - \frac{\pi}{\bar{\lambda} z_2}(\rho_2^{\,2} + \rho_1^{\,2})$. If $I_1$ and $I_2$ are constant and independent of $(x, y)$, then




$$V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}\mu_{12} \overset{I_1 = I_2}{=} \mu_{12},$$ which is constant across the observation region.

## 5.4 Cross-Spectral Purity




Let us consider again the Young's two-beam interference experiment shown above. For the known normalized power spectrum at $P_1$ and $P_2$, it is interesting to ask *what* *is the shape of the power spectral density of the resultant light at Q*? To answer the question, first we have to learn more some useful concepts.

### 5.4.1 Power Spectrum of the Superposition of Two Light Beams

We can write the analytic signal at $Q$ as

$$u(Q,t) = K_1 u(P_1, t-\tau_1) + K_2 u(P_2, t-\tau_2)\,.$$

Assuming the two waves to be superimposed at $Q$ have the same power spectrum $\mathcal{P}_1(\nu) = \mathcal{P}_2(\nu) = \mathcal{P}(\nu)$, but suffer time delays $\tau_1$ *and* $\tau_2$.

The self coherence function at $Q$ can be deduced with

$$\begin{aligned}\Gamma_Q(\tau) &= \langle u(Q,t+\tau)u^*(Q,t)\rangle \\ &= \langle [K_1 u(P_1,t+\tau-\tau_1) + K_2 u(P_2,t+\tau-\tau_2)][K_1 u(P_1,t-\tau_1) + K_2 u(P_2,t-\tau_2)]^* \rangle\end{aligned}.$$

By using the notation of $\Gamma_{12}(\tau) = \langle u(P_1,t+\tau)u^*(P_2,t)\rangle = \langle u_1(t+\tau)u_2^*(t)\rangle$, we can rewrite $\Gamma_Q(\tau)$ as

$$\Gamma_Q(\tau) = K_1^2\,\Gamma_{11}(\tau) + K_2^2\,\Gamma_{22}(\tau) + K_1K_2\,\Gamma_{12}(\tau_2-\tau_1+\tau) + K_1K_2\,\Gamma_{21}(\tau_1-\tau_2-\tau)\,.$$

Then normalize the self coherence function at $Q$

$$\begin{aligned}\gamma_Q(\tau) = \Gamma_Q(\tau)/\Gamma_Q(0) &= \frac{K^2\,\Gamma_{11}(\tau) + K^2\,\Gamma_{22}(\tau) + 2K^2\,\mathrm{Re}[\Gamma_{12}(\tau_2-\tau_1+\tau)]}{K^2 I_1 + K^2 I_2 + 2K^2\,\mathrm{Re}[\Gamma_{12}(\tau_2-\tau_1)]} \\ &= \frac{\gamma_{11}(\tau) + A\cdot\mathrm{Re}[\gamma_{12}(\tau_2-\tau_1+\tau)]}{1 + A\cdot\mathrm{Re}[\gamma_{12}(\tau_2-\tau_1)]}\end{aligned},$$

where $A = 2\sqrt{I_1 I_2}\big/(I_1+I_2)$.

We then perform Fourier transform (from τ to ν) on the above equation to yield the power spectrum at $Q$

$$\hat{\mathcal{P}}_Q(\nu) = \frac{\hat{\mathcal{P}}(\nu) + A \cdot \mathrm{Re}[\hat{\mathcal{P}}_{12}(\nu) e^{-j2\pi\nu(\tau_2 - \tau_1)}]}{1 + A \cdot \mathrm{Re}[\gamma_{12}(\tau_2 - \tau_1)]} .$$

### 5.4.2 Cross-Spectral Purity and Reducibility

We may ask: *at what conditions the power spectral density of two-superimposed-beam is equal to the original beam* $\hat{\mathcal{P}}_Q(\nu) = \hat{\mathcal{P}}(\nu)$ ?

Let us calculate the difference

$$\hat{\mathcal{P}}_Q(\nu) - \hat{\mathcal{P}}(\nu) = \frac{A \cdot \mathrm{Re}[\hat{\mathcal{P}}_{12}(\nu) e^{-j2\pi\nu(\tau_2 - \tau_1)} - \gamma_{12}(\tau_2 - \tau_1)\hat{\mathcal{P}}(\nu)]}{1 + A \cdot \mathrm{Re}[\gamma_{12}(\tau_2 - \tau_1)]} .$$

If $\hat{\mathcal{P}}_Q(\nu) = \hat{\mathcal{P}}(\nu)$ for all $I_2$, $I_1$ and $\tau_2 - \tau_1$, then the nominator shall be zero, which leads to

$$\hat{\mathcal{P}}_{12}(\nu) e^{-j2\pi\nu(\tau_2 - \tau_1)} = \gamma_{12}(\tau_2 - \tau_1)\hat{\mathcal{P}}(\nu) .$$

Since this equality must hold for all $\tau_2 - \tau_1$ (that is $u(P_1, t - \tau_1)$ *and* $u(P_2, t - \tau_2)$ are uncorrelated for all $\tau_2 - \tau_1$) $\Rightarrow$ $\hat{\mathcal{P}}_{12}(\nu) = 0$ *and* $\gamma_{12}(\tau_2 - \tau_1) = 0$. This condition is very strict and cannot be valid for most cases. To relax the condition, let us restrict $\tau$ by expressing

- $\tau_2 - \tau_1 = \tau_o + \Delta\tau$ *with* $\Delta\tau \ll \dfrac{1}{\Delta\nu} = \tau_c$; and
- Note $\hat{\mathcal{P}}_{12}(\nu) \neq 0$ for $-\dfrac{\Delta\nu}{2} < \nu = \bar{\nu} + \delta\nu < \dfrac{\Delta\nu}{2}$.

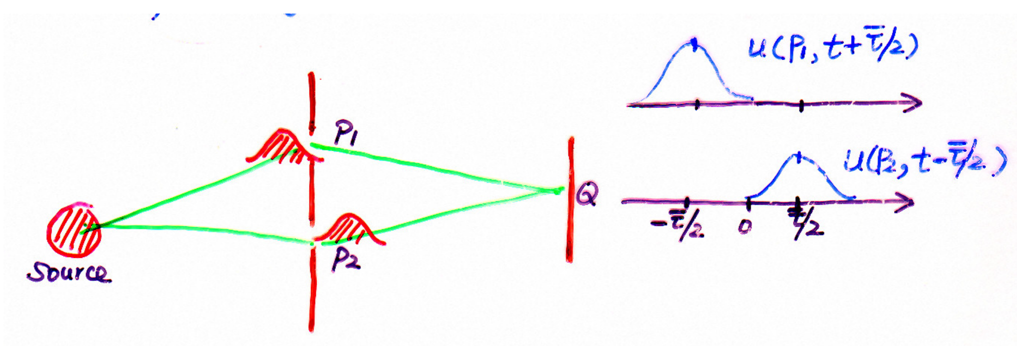
Therefore

$$\gamma_{12}(\tau_o+\Delta\tau)=\int_0^\infty \hat{\mathcal{P}}_{12}(\nu)e^{-j2\pi\nu(\tau_o+\Delta\tau)}d\nu=\int_0^\infty \hat{\mathcal{P}}_{12}(\nu)e^{-j2\pi(\bar{\nu}+\delta\nu)(\tau_o+\Delta\tau)}d\nu$$
$$=\int_0^\infty \hat{\mathcal{P}}_{12}(\nu)e^{-j2\pi\bar{\nu}(\tau_o+\Delta\tau)}\cdot e^{-j2\pi\delta\nu\tau_o}\cdot e^{-j2\pi\delta\nu\Delta\tau}d\nu$$
$$\xrightarrow{\Delta\nu\cdot\Delta\tau\ll 1} e^{-j2\pi\bar{\nu}\cdot\Delta\tau}\int_0^\infty \hat{\mathcal{P}}_{12}(\nu)\cdot e^{-j2\pi(\bar{\nu}+\delta\nu)\tau_o}d\nu \quad .$$

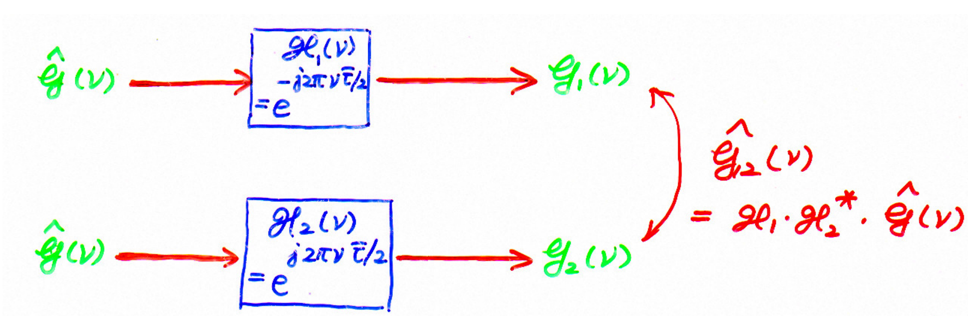
$$\gamma_{12}(\tau_o+\Delta\tau)\simeq e^{-j2\pi\bar{\nu}\cdot\Delta\tau}\cdot\int_0^\infty \hat{\mathcal{P}}_{12}(\nu)e^{-j2\pi\nu\cdot\tau_o}d\nu=\gamma_{12}(\tau_o)\cdot e^{-j2\pi\bar{\nu}\cdot\Delta\tau} .$$

Therefore, $\hat{\mathcal{P}}_{12}(\nu)e^{-j2\pi\nu(\tau_0+\Delta\tau)}=\gamma_{12}(\tau_0+\Delta\tau)\hat{\mathcal{P}}(\nu)=\gamma_{12}(\tau_0)e^{-j2\pi\bar{\nu}\Delta\tau}\hat{\mathcal{P}}(\nu)$. That is $\hat{\mathcal{P}}_{12}(\nu)e^{-j2\pi\nu\tau_0}=\gamma_{12}(\tau_0)\hat{\mathcal{P}}(\nu)$, which is oscillatory in $\nu$. This equality can hold if we choose $\tau_0$ properly.

Considering the field $u(P_1)$ and $u(P_2)$ from the same source to the screen are temporally shifted by $\bar{\tau}$,

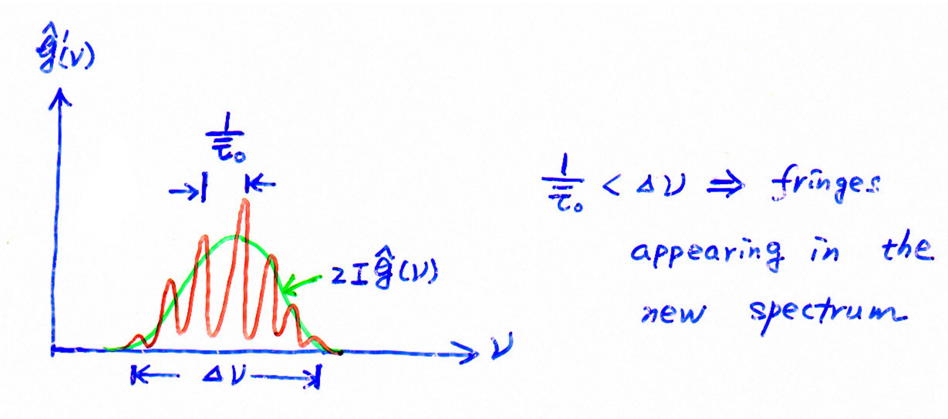



Then from the cross-spectral density of two linearly filtered random processes,



$\therefore \quad \hat{\mathcal{P}}_{12}(\nu) = \hat{\mathcal{P}}(\nu) e^{j2\pi\nu\overline{\tau}}$ for the two relatively delayed wave disturbances. The normalized spectrum of the superposition of two light beams that are identical except for a relative delay $\overline{\tau}$ becomes

$$\hat{\mathcal{P}}'(\nu) = \hat{\mathcal{P}}_1(\nu) + \hat{\mathcal{P}}_2(\nu) + \hat{\mathcal{P}}_{12}(\nu) + \hat{\mathcal{P}}_{21}(\nu) = \hat{\mathcal{P}}(\nu)[1 + \cos 2\pi\nu\overline{\tau}].$$



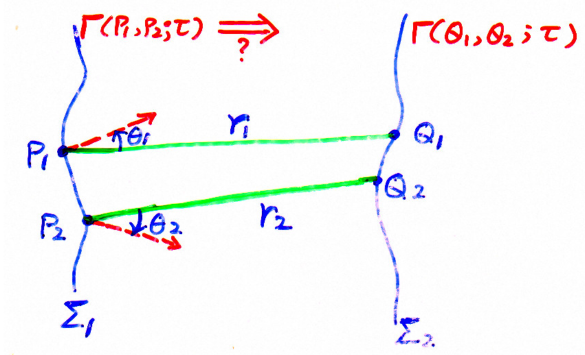

$$FT^{-1}\{\hat{\mathcal{P}}_{12}(\nu) e^{j2\pi\nu\tau_0}\} = FT^{-1}\{\hat{\mathcal{P}}(\nu)\}\gamma_{12}(\tau_0)$$

$$\Rightarrow \quad \gamma_{12}(\tau + \tau_0) = \gamma_{12}(\tau_0) \cdot \gamma(\tau).$$

Here the first term $\gamma_{12}(\tau_0)$ is due to a difference in optical pathlength, reflecting the spatial coherence. This equation decomposes the complex degree of coherence $\gamma_{12}(\tau + \tau_0)$ into a product of a spatial coherence and a temporal coherence $\gamma(\tau)$.

For two beams to be cross spectrally pure, all spectral components of one beam must have *the same normalized cross-correlation* with the corresponding spectral components of the other beam. That is under the quasi-monochromatic condition, we have $\gamma_{12}(\tau + \tau_0) = \gamma_{12}(\tau_0) \cdot \gamma(\tau) = \mu_{12} \cdot \gamma(\tau)$.

## 5.5 Propagation of Mutual Coherence




In this section, we would like study on *how the mutual coherence propagates from a surface with given known distribution* $\Gamma(P_1, P_2; \tau)$.

By invoking the Huygens-Fresnel principle with narrowband light, we have

$$u(Q_1, t+\tau) = \iint_{\Sigma_1} \frac{\chi(\theta_1)}{j\bar{\lambda}\, r_1} u(P_1, t+\tau-\frac{r_1}{c})\, dS_1\,, \quad u(Q_2, t) = \iint_{\Sigma_1} \frac{\chi(\theta_2)}{j\bar{\lambda}\, r_2} u(P_2, t-\frac{r_2}{c})\, dS_2\,.$$

The *mutual coherence* on $\Sigma_1$ is known by $\Gamma(P_1, P_2; \tau) = < u(P_1, t+\tau) u^*(P_2, t) >$.

The mutual coherence on $\Sigma_2$ is by definition

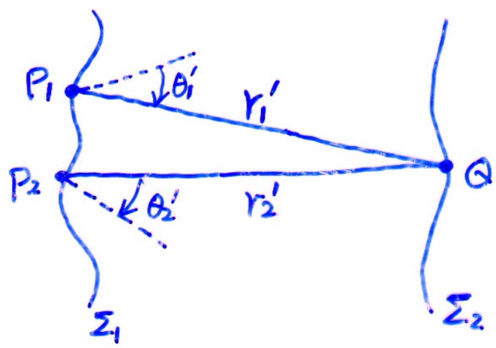
$$\begin{aligned} \Gamma(Q_1, Q_2; \tau) &= < u(Q_1, t+\tau) u^*(Q_2, t) > \\ &= \iint_{\Sigma_1} \frac{\chi(\theta_1)}{\bar{\lambda} r_1} \iint_{\Sigma_1} \frac{\chi(\theta_2)}{\bar{\lambda} r_2} < u(P_1, t+\tau-\frac{r_1}{c}) u^*(P_2, t-\frac{r_2}{c}) > dS_2\, dS_1 \\ &= \iint_{\Sigma_1} \frac{\chi(\theta_1)}{\bar{\lambda} r_1} \iint_{\Sigma_1} \frac{\chi(\theta_2)}{\bar{\lambda} r_2} \cdot \Gamma(P_1, P_2; \tau + \frac{r_2 - r_1}{c}) dS_2\, dS_1 \end{aligned}.$$

The mutual intensity on $\Sigma_2$ for the quasi-monochromatic conditions can be obtained from $\Gamma(Q_1, Q_2; 0)$

$$
\begin{aligned}
J(Q_1, Q_2) &= \Gamma(Q_1, Q_2; 0) \\
&= \iint_{\Sigma_1} \frac{\chi(\theta_1)}{\bar{\lambda} r_1} [\iint_{\Sigma_1} \frac{\chi(\theta_2)}{\bar{\lambda} r_2} \Gamma(P_1, P_2; \frac{r_2 - r_1}{c}) dS_2] \, dS_1 \\
&= \iint_{\Sigma_1} \frac{\chi(\theta_1)}{\bar{\lambda} r_1} [\iint_{\Sigma_1} \frac{\chi(\theta_2)}{\bar{\lambda} r_2} J(P_1, P_2) \, e^{-2\pi j (r_2 - r_1)/\bar{\lambda}} dS_2] \, dS_1
\end{aligned}
.
$$

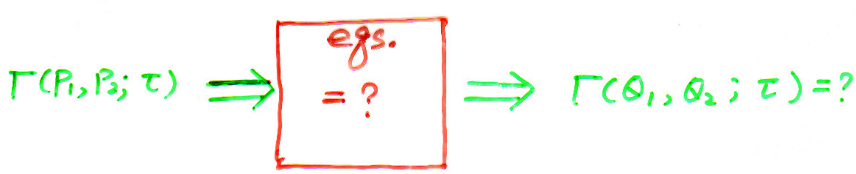
The intensity distribution on the surface $\Sigma_2$ can be found by letting $Q_2 \to Q_1$

$$
I(Q) = J(Q_1 = Q_2 = Q) = \iint_{\Sigma_1} \frac{\chi(\theta_1')}{\bar{\lambda} r_1'} [\iint_{\Sigma_1} \frac{\chi(\theta_2')}{\bar{\lambda} r_2'} \cdot J(P_1, P_2) \, e^{-2\pi j (r_2' - r_1')/\bar{\lambda}} dS_2] \, dS_1 .
$$




### 5.5.1 Propagation of Mutual Coherence

We have derived a mathematical expression of the mutual coherence on an arbitrary surface. But can we find equations such that by solving these equations with a given $\Gamma(P_1, P_2; \tau)$, we can obtain $\Gamma(Q_1, Q_2; \tau)$ ?




Note in free space, the analytic wave disturbance $u(P, t)$ satisfies:

$$\nabla^2 u(P,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(P,t) = 0 .$$

From $\Gamma_{12}(\tau) =< u(P_1, t+\tau) u^*(P_2, t) >=< u_1(t+\tau) u_2^*(t) >$, we obtain

$$\begin{aligned}
\nabla_1{}^2 \Gamma_{12}(\tau) &= \left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y_1{}^2} + \frac{\partial^2}{\partial z_1{}^2}\right) < u(x_1, y_1, z_1; t+\tau) u^*(x_2, y_2, z_2; t) > \\
&=< [\nabla_1{}^2 u_1(t+\tau)] \cdot u_2{}^*(t) >=< \frac{1}{c^2}\frac{\partial^2 u_1(t+\tau)}{\partial \tau^2} \cdot u_2{}^*(t) > \\
&= \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} < u_1(t+\tau) u_2{}^*(t) >
\end{aligned} .$$

Therefore, we derive the propagation equations for $\Gamma_{12}(\tau)$

$$\nabla_1{}^2 \Gamma_{12}(\tau) = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau); \qquad \nabla_2{}^2 \Gamma_{12}(\tau) = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau) .$$

Note that $J_{12}(\tau) = J_{12}\, e^{-j\frac{2\pi c}{\bar{\lambda}}\tau}$, therefore

$$\nabla_1{}^2 J_{12}(\tau) + \left(\frac{2\pi}{\bar{\lambda}}\right)^2 J_{12} = \nabla_1{}^2 J_{12}(\tau) + \bar{k}^2 J_{12} = 0 .$$

The forms of the mutual coherence and mutual intensity obtained from HF principle are a special solution of the above propagation equation.

### 5.5.2 Propagation of Cross-Spectral Density

Consider $\Gamma_{12}(\tau) \xrightarrow{FT} \hat{\mathcal{P}}_{12}(\nu)$ , *what is the propagation equation* of $\hat{\mathcal{P}}_{12}(\nu)$ ?

From $(\nabla_1{}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2})\Gamma_{12}(\tau) = \int_0^\infty (\nabla_1{}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2})\, \hat{\mathcal{P}}_{12}(\nu)\, e^{-j2\pi\nu\tau} d\nu = 0$ for all $\tau$ and $\hat{\mathcal{P}}_{12}(\nu)$ .

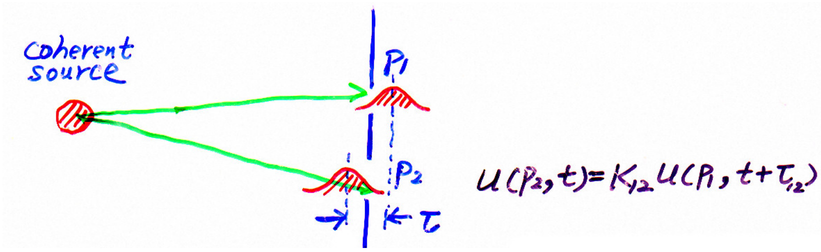
Therefore, $\nabla_1{}^2\hat{\mathcal{P}}_{12}(\nu)+\left(\frac{2\pi\nu}{c}\right)^2\hat{\mathcal{P}}_{12}(\nu)=0$, implying that cross-spectral densities shall obey the same propagation laws as do mutual intensities. To find the solution for cross-spectral density, the corresponding result for mutual intensity can be used, subject only to the requirement that the parameter $\bar{k}=\frac{2\pi}{\bar{\lambda}}=\frac{2\pi\bar{\nu}}{c}$ must be replaced by $k=\frac{2\pi\nu}{c}$.

## 5.6 Limiting Forms of the Mutual Coherence Function

### 5.6.1 A Coherent Field

A wave field is called fully coherent, if for every pair of points ($P_1$, $P_2$) on the wavefront, there exists a delay τ depending on ($P_1$, $P_2$), such that $|\gamma_{12}(\tau)|=1$. That is $\max_{\tau}|\gamma_{12}(\tau)|=1$ for every pair of ($P_1$, $P_2$). Note $|\gamma_{12}(\tau)|=\frac{|<u(P_1,t+\tau)u^*(P_2,t)>|}{\sqrt{<|u(P_1,t+\tau)|^2><|u(P_2,t)|^2>}}$.

Let $u(P,t)=A(P,t)\,e^{-j2\pi\bar{\nu}t}$, $|\gamma_{12}(\tau)|=\frac{|<A(P_1,t+\tau)A^*(P_2,t)>|}{\sqrt{<|A(P_1,t+\tau)|^2><|A(P_2,t)|^2>}}\le 1$. Thus, $|\gamma_{12}(\tau)|=1$ if and only if $A(P_2,t)=K_{12}A(P_1,t+\tau)$. That is a wave field is called perfectly coherent *if and only if for every pair of ($P_1$, $P_2$) there exists a time delay τ such that the complex envelopes of the two wavefronts*, relatively delayed by the required τ, *differ by only a time-independent complex constant*.




If the quasi-monochromatic conditions are imposed, the same $\tau_{12} = \tau$ should be applied for any pair of ($P_1$, $P_2$). Now by letting $P_1 \to P_2$

$$\lim_{P_1 \to P_2} |\Gamma_{12}(\tau)| = \max_{\tau} |\Gamma_{12}(\tau)| = 1 \quad \Rightarrow \quad \tau = 0$$
$$\therefore \quad A(P_2,t) = K_{12}A(P_1,t) \qquad ,$$

Implying that for a quasi-monochromatic fully coherent light, the complex envelopes at all points vary in unison differing from each other only by time-invariant amplitude and phase factor, i.e., $A(P_1,t) = \dfrac{A(P_1)}{\sqrt{I(P_0)}} A(P_0,t)$ and $A(P_2,t) = \dfrac{A(P_2)}{\sqrt{I(P_0)}} A(P_0,t)$. Therefore,

$$J_{12} = < A(P_1,t)A^*(P_2,t) > = \frac{A(P_1)A^*(P_2)}{I(P_0)} \cdot < A(P_0,t) \cdot A^*(P_0,t) > = A(P_1)A^*(P_2),$$

and $\mu_{12} = \dfrac{J_{12}}{\sqrt{J_{11}J_{22}}} = \dfrac{A(P_1)}{|A(P_1)|} \cdot \dfrac{A^*(P_2)}{|A(P_2)|} = e^{j[\Phi(P_2) - \Phi(P_1)]}$.

The fringe pattern generated by a Young's experiment becomes

$$I(Q) = I_1 + I_2 + 2\sqrt{I_1 I_2}\, \mu_{12} \cos[\frac{2\pi}{\lambda}(r_2 - r_1)] = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos[\frac{2\pi}{\lambda}(r_2 - r_1) + \Phi(P_2) - \Phi(P_1)].$$

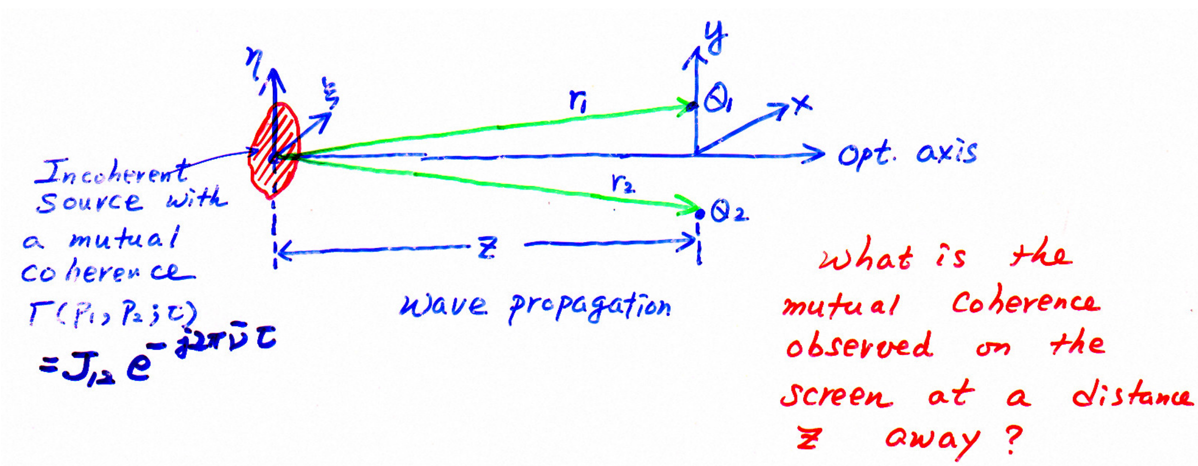
$V = \dfrac{I_{\max} - I_{\min}}{I_{\max} I_{\min}} = 1$ for a fully coherent light.

### 5.6.2 An Incoherent Field

For an incoherent field, the mutual intensity shall become

$J(P_1,P_2) = \langle A(P_1,t)\cdot A^*(P_2,t)\rangle = K\,I(P_1)\delta(x_1-x_2, y_1-y_2)$, where $K = \frac{(\bar{\lambda})^2}{\pi}$ denting a dimension of squared length.

## 5.7 The Van Citter-Zernike Theorem




From $\Gamma(Q_1, Q_2; \tau)$ and let $\tau = 0$

$$\begin{aligned} J(Q_1, Q_2) &= \Gamma(Q_1, Q_2; 0) \\ &= \iint_{\Sigma_1} \frac{\chi(\theta_1)}{\bar{\lambda} r_1} [\iint_{\Sigma_1} \frac{\chi(\theta_2)}{\bar{\lambda} r_2} \cdot J(P_1, P_2)\, e^{-2\pi j (r_2 - r_1)/\bar{\lambda}} dS_2 ] dS_1 \,. \end{aligned}$$

For an incoherent source with $J(P_1, P_2) = K\, I(P_1)\, \delta(|P_1 - P_2|)$,

$$J(Q_1, Q_2) = \frac{K}{(\bar{\lambda})^2} \iint_{\Sigma} \frac{\chi(\theta_1)}{r_1} \frac{\chi(\theta_2)}{r_2} I(P_1) e^{-2\pi j (r_2 - r_1)/\bar{\lambda}} dS \,.$$

Note

■ at far field $\frac{1}{r_1}\frac{1}{r_2} \simeq \frac{1}{z^2}$, and

■ $\chi(\theta_1) \sim \chi(\theta_2) \sim 1$ for small angle

$$\therefore \quad J(Q_1, Q_2) = \frac{K}{(\bar{\lambda} z)^2} \iint_{\Sigma} I(P_1) e^{-2\pi j (r_2 - r_1)/\bar{\lambda}} dS .$$

■ By invoking the paraxial approximation in the phase factors,

$$r_1 \simeq z + \frac{(x_1 - \xi)^2 + (y_1 - \eta)^2}{2z}, \quad r_2 \simeq z + \frac{(x_2 - \xi)^2 + (y_2 - \eta)^2}{2z} .$$

Let $Q_1 = (x_1, y_1), Q_2 = (x_2, y_2)$ and define $\Delta x = x_2 - x_1, \Delta y = y_2 - y_1$, we obtain the Van Citter-Zernike theorem

$$J(Q_1, Q_2) = \frac{K\, e^{-j\Psi}}{(\bar{\lambda} z)^2} \iint_{\Sigma} I(\xi, \eta) e^{j\frac{2\pi}{\bar{\lambda} z}(\Delta x \cdot \xi + \Delta y \cdot \eta)} d\xi\, d\eta$$

where $\Psi = \frac{\pi}{\bar{\lambda} z}[\rho_2{}^2 - \rho_1{}^2]$ denoting a normalized squared length difference from $Q_1$ and $Q_2$ to the optical axis. The corresponding *complex coherence factor* becomes

$$\mu(Q_1, Q_2) = \frac{e^{-j\Psi} \iint_{\Sigma} I(\xi, \eta) e^{j\frac{2\pi}{\bar{\lambda} z}(\Delta x \cdot \xi + \Delta y \cdot \eta)} d\xi\, d\eta}{\iint_{\Sigma} I(\xi, \eta) d\xi\, d\eta} = \frac{\iint_{\Sigma} I(\xi, \eta) e^{j\frac{2\pi}{\bar{\lambda} z}(\Delta x \cdot \xi + \Delta y \cdot \eta)} d\xi\, d\eta}{\iint_{\Sigma} I(\xi, \eta) d\xi\, d\eta} .$$

For $e^{-j\Psi} \sim 1$ (*i.e.*, when $\Psi = \frac{\pi}{\bar{\lambda} z}(\rho_2{}^2 - \rho_1{}^2) \simeq 0$), which can occur at

i. $z \gg \frac{2}{\bar{\lambda}}(\rho_2{}^2 - \rho_1{}^2)$ for far field; or

ii. $\rho_2 \sim \rho_1$ for symmetrical positions relative to the optical axis;

iii. $Q_1$ and $Q_2$ are on a spherical surface instead of on the planar screen.

We can deduce a coherent area of the light source on the *xy* plane as

$A_c = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|\mu(\Delta x, \Delta y)\right|^2 d(\Delta x) d(\Delta y) = \frac{(\bar{\lambda} z)^2}{A_s}$, where $A_s$ is the source area of a uniformly bright incoherent source.

### 5.7.2 An Application Example of Van Citter-Zernike Theorem

Let $I(\xi,\eta) = I_0\, circ(\sqrt{\xi^2 + \eta^2}/a)$ be a uniformly bright source with a diameter of 2*a*. The resulting mutual intensity after propagation becomes

$$J(Q_1, Q_2) = \frac{K}{(\bar{\lambda} z)^2} \iint_{\Sigma} I(\xi,\eta) e^{j\frac{2\pi}{\bar{\lambda} z}(\Delta x \cdot \xi + \Delta y \cdot \eta)} d\xi\, d\eta = \frac{\pi a^2 K I_0}{(\bar{\lambda} z)^2} \cdot \left\{\frac{2J_1(u)}{u}\right\},$$

where $u = \frac{2\pi\, a}{\bar{\lambda} z}\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{2\pi\, a}{\bar{\lambda} z} S$.

For $J_1(u) = 0 \quad \Rightarrow \quad u_1 = 3.83$,

$\frac{2\pi}{\bar{\lambda}}\frac{a}{z} S_0 = 3.83 \quad \Rightarrow \quad \frac{\pi}{\bar{\lambda}} \cdot \frac{2a}{z} \cdot S_0 = \frac{\pi}{\bar{\lambda}} \cdot \theta \cdot S_0 = 3.83$.

We then derive the Rayleigh-Abbe resolution criteria as $S_0 = 1.22\frac{\bar{\lambda}}{\theta}$, and note $A_c = \frac{\bar{\lambda}^2 z^2}{A_s} = \frac{\bar{\lambda}^2 z^2}{\pi a^2} \simeq \frac{\bar{\lambda}^2}{\Omega_s}$, implying that *a coherent spot can be obtained even* *with an incoherent source*. The resulting coherent area is inversely proportional to the solid angle extended by the incoherent source at the observation position.