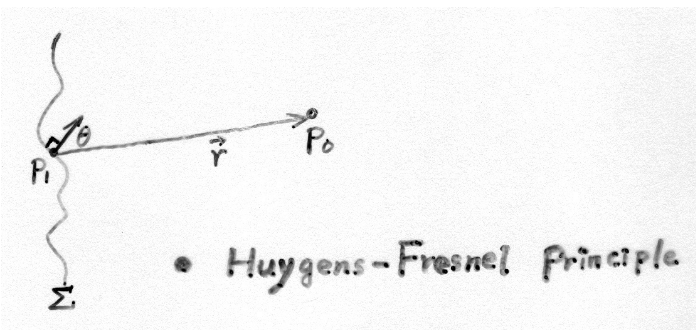
Chapter 4 Statistical Properties of Light

4.1 Propagation of Monochromatic Light

Consider $u(P,t)$ to be the amplitude (at a position P and time *t*) of one polarization component of the electric field of a monochromatic optical wave

$$u(P,t) \equiv U(P,\nu)\, e^{-j2\pi\nu t}.$$

From Huygens-Fresnel principle, the amplitude at P_o is contributed from **each point on Σ**, which acts as new source with a strength of $U(P_1,\nu)/(j\lambda)$ and radiates a spherical wave (e^{jkr}/r) with a directional factor $\chi(\theta)$ (*i.e.*, the obliquity factor of radiation process with $0 \le |\chi(\theta)| \le 1$):



$$U(P_o,\nu) = \frac{1}{j\lambda}\int_{\Sigma} U(P_1,\nu)\cdot\frac{e^{jkr}}{r}\cdot\chi(\theta)\,d\Sigma \quad ; r \gg \lambda.$$

4.2 Propagation of Non-monochromatic Light

Let $u_T(P,t)$ be an analytic signal, truncating from $u(P,t)$ to the interval $(-T/2, T/2)$ to ensure the existence of its Fourier transform. Thus,

$$u_T(P,t) \equiv \int_0^{\infty} \mathbf{2}U_T(P,\nu)\, e^{-j2\pi\nu t} d\nu \ ,$$

where $U_T(P,\nu)$ is the Fourier transform of the **real signal** $u_T(P,t)$.

Now by invoking the Huygens-Fresnel principle to derive the amplitude at $\boldsymbol{P}_o$

$$
\begin{aligned}
u(P_o,t) &= \lim_{T\to\infty} u_T(P_o,t) = \lim_{T\to\infty}\int_0^\infty 2U_T(P_o,\nu)\cdot e^{-j2\pi\nu t}\,d\nu \\
&= \lim_{T\to\infty}\int_0^\infty 2\left\{\int_\Sigma \frac{U_T(P_1,\nu)}{j\lambda}\cdot\frac{e^{jkr}}{r}\cdot\chi(\theta)\,d\Sigma\right\}\cdot e^{-j2\pi\nu t}d\nu
\end{aligned}
$$

By exchanging the order of integration and recall $k = 2\pi/\lambda = 2\pi\nu/c$

$$
\begin{aligned}
u(P_o,t) &= \lim_{T\to\infty}\int_0^\infty 2\left\{\int_\Sigma \frac{U_T(P_1,\nu)}{j\lambda}\cdot\frac{e^{jkr}}{r}\cdot\chi(\theta)\,d\Sigma\right\}\cdot e^{-j2\pi\nu t}d\nu \\
&= \lim_{T\to\infty}\int_\Sigma \frac{2\chi(\theta)}{2\pi cr}\cdot\left\{\int_0^\infty[-j2\pi\nu\, U_T(P_1,\nu)]e^{-j2\pi\nu(t-r/c)}d\nu\right\}d\Sigma \\
&= \lim_{T\to\infty}\int_\Sigma \frac{\frac{d}{dt}u_T(P_1,t-r/c)}{2\pi cr}\cdot\chi(\theta)\,d\Sigma
\end{aligned}
$$

$u(P_o,t) = \lim_{T\to\infty}\int_\Sigma \frac{\frac{d}{dt}u_T(P_1,t-r/c)}{2\pi cr}\cdot\chi(\theta)\,d\Sigma$ can serve as the equation to describe the propagation of **non-monochromatic light**.

4.2.1 Narrowband Light

For a narrowband light that meets the criterion of $\Delta\nu \ll \bar{\nu}$, where $\Delta\nu$ and $\bar{\nu}$ denote the bandwidth and the central frequency of the spectrum,

$$
\begin{aligned}
u_T(P_o,t) &= \int_\Sigma \frac{1}{jcr}\cdot\left\{\int_0^\infty[2\nu\, U_T(P_1,\nu)]e^{-j2\pi\nu(t-r/c)}d\nu\right\}\chi(\theta)\,d\Sigma \\
&\simeq \int_\Sigma \frac{\bar{\nu}}{jcr}\cdot\left\{\int_0^\infty 2U_T(P_1,\nu)e^{-j2\pi\nu(t-r/c)}d\nu\right\}\chi(\theta)d\Sigma
\end{aligned}
$$

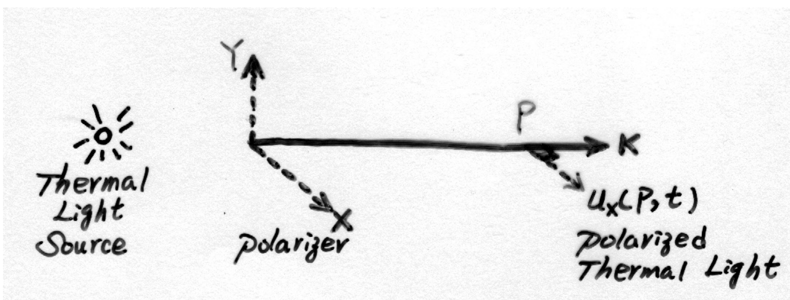
Note that the integration over ν is conducted over the frequency range of $\Delta\nu$ where $U_T(P,\nu)$ is nonzero, therefore we can let T taking the limit $T \to \infty$, which leads to the following result

$$u_T(P_o,t) \overset{T\to\infty}{\to} \int_\Sigma \frac{1}{j\bar{\lambda}r} \cdot u(P_1,t-r/c)\,\chi(\theta)\,d\Sigma \quad ; \quad r \gg \bar{\lambda}.$$

4.3 Polarized and Unpolarized Thermal Light

■ Polarized Thermal Light

Referring to the following diagram,



the amplitude of X polarization component of an electric field radiated from an ensemble of atoms can be expressed as

$$u_X(P,t) \equiv \sum_{i \in \text{all atoms}} u_i(P,t) \;.$$

We can view a thermal light source as a large number of independent random radiators. Therefore, according to the central limit theorem, $u_X(P,t)$ shall obey the Gaussian random process.

Let $A_X(P,t) \equiv u_X(P,t)e^{j2\pi\bar{\nu}t}$ denotes a complex envelope of a light field with $\bar{\nu}$ the center frequency of the wave. Then

$$A_X(P,t) \equiv \sum_{i \in all\ atoms} A_i(P,t) = u_X(P,t)e^{j2\pi\bar{\nu}t}.$$

Owing to that the arrival time of the radiation from a particular atom is totally unpredictable, the phase of that radiation shall be uniformly distributed on the primary interval, *i.e.*, the phases of the $A_i(P,t)$ are

- statistical independent; and
- uniformly distributed on $[-\pi, \pi]$.

The various contributions in $A_X(P,t) \equiv \sum_{i \in all\ atoms} A_i(P,t)$ and u_X are randomly phased and independent, leading to that both $A_X(P,t)$ and $u_X(P,t)$ belong to circular complex Gaussian random processes. That is

- $\mathrm{Re}(A_X)$ and $\mathrm{Im}(A_X)$ are independent, and are
- identically distributed zero-mean Gaussian random variables.

Photodetectors response to light intensity instead of field strength,

$I_X(P,t) \equiv |u_X(P,t)|^2 = |A_X(P,t)|^2 =$ instantaneous intensity,

$I_X(P) \equiv < I_X(P,t) > = \bar{I}_X(P) =$ average intensity.

Note $I_X(P,t)$ is a random process, which is the squared length of a random phasor sum

$$I_X(P,t) = \left| \sum_{i \in all\ atoms} u_{i,X}(P,t)e^{-j2\pi\nu t} \right|^2.$$

Let $A \equiv |A_X(P,t)|$ and $I \equiv I_X(P,t)$ and note that A obeys a probability law of

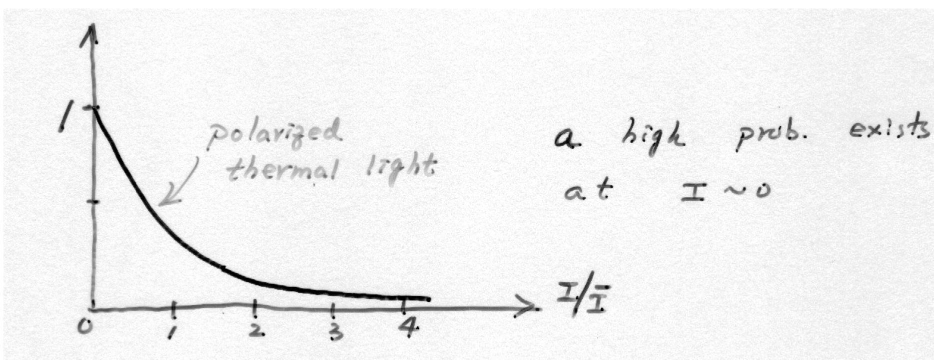
$$p_A(A) = \begin{cases} \dfrac{A}{\sigma^2} e^{-A^2/(2\sigma^2)}; & A \geq 0 \\ 0\,; & elsewhere \end{cases}.$$

Take a transformation of $I = A^2$, $A = \sqrt{I}$, we then obtain

$$p_I(I) = p_A(\sqrt{I}) \cdot \left| \frac{dA}{dI} \right| = \begin{cases} \dfrac{1}{2\sigma^2} e^{-I/(2\sigma^2)}; & I \geq 0 \\ 0; & elsewhere \end{cases}.$$

Let $\sigma_I = \overline{I} = 2\sigma^2$,

$$p_I(I) = \begin{cases} \dfrac{1}{\overline{I}} e^{-I/\overline{I}}; & I \geq 0 \\ 0\,; & elsewhere \end{cases},$$



It implies instantaneous intensity $I(P,t)$ obeys a negative exponential probability law with a high probability at small intensity $I(P,t) \simeq 0$.

■ Unpolarized Thermal Light

An appropriate definition of unpolarized thermal light can be defined to possess the following two characteristics

- Intensity of the light passed by an analyzer is independent of the rotational angle of the analyzer;

- $< u_X(P,t+\tau)u_Y^*(P,t) >$ is identically zero for all rotational orientation of the X-Y coordinate axes and for all delay τ.

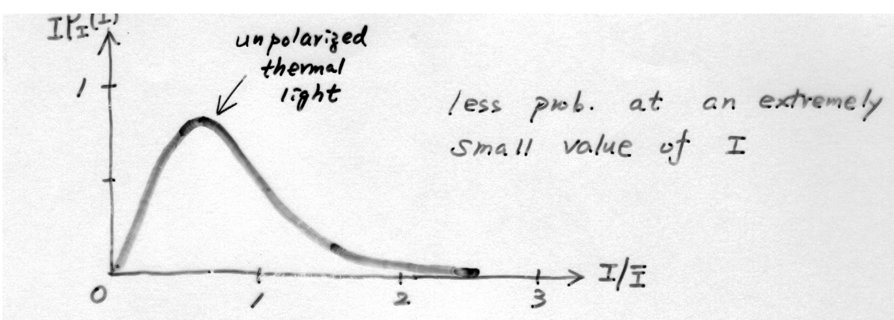
From the above conditions, we get $u_X(P,t)$ and $u_Y(P,t)$ are statistically independent circular complex Gaussian random processes, which are uncorrelated for all time delay τ.

$$\begin{aligned} I(P,t) &= | u_X(P,t) + u_Y(P,t) |^2 \\ &= | u_X(P,t) |^2 + | u_Y(P,t) |^2 \\ &= I_X(P,t) + I_Y(P,t) \end{aligned}.$$

In the previous section, we have verified that $I_X(P,t)$ and $I_Y(P,t)$ obey negative exponential statistics with $\bar{I}_X = \bar{I}_Y = \bar{I}/2$. Therefore,

$$p_I(I) = \int_{-\infty}^{\infty} p_{I_X}(I - I_X) p_{I_Y}(I_Y)\, dI_Y$$

$$= \begin{cases} \int_o^I (\frac{2}{\bar{I}})^2 \cdot e^{-2\xi/\bar{I}} \cdot e^{-2(I-\xi)/\bar{I}} d\xi = (\frac{2}{\bar{I}})^2 \cdot I \cdot e^{-2I/\bar{I}} ; & I \geq 0 \\ 0 ; & elsewhere \end{cases}.$$



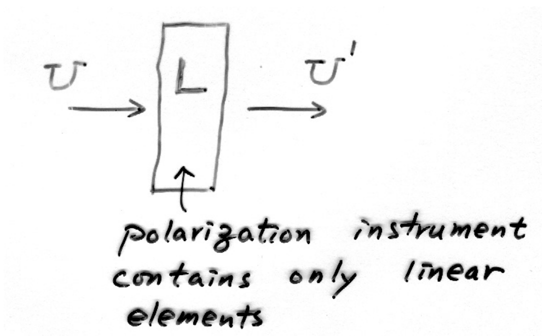
4.4 Partially Polarized Thermal Light

To specify the state of a narrowband light field at position P and time t, we shall invoke the formalism of Jones matrices by defining

$$U(P,t)=\begin{bmatrix} u_X(P,t) \\ u_Y(P,t) \end{bmatrix}.$$

The effect of a linear optical device on the incident light field can be depicted with

$$U'(P',t)=\begin{bmatrix} u_X'(P',t) \\ u_Y'(P',t) \end{bmatrix}=LU=\begin{bmatrix} L_{11} & L_{12} \\ L_{13} & L_{14} \end{bmatrix}\begin{bmatrix} u_X(P,t) \\ u_Y(P,t) \end{bmatrix}.$$

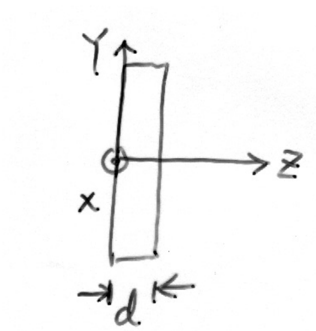


(i) If a device produces a rotation of the X-Y coordinate system, then

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{13} & L_{14} \end{bmatrix}=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

(ii) Retardation Plate

The optical operation $\boldsymbol{L}$ on the incident field by means of a birefringent material can be found to be



$$\begin{bmatrix} L_{11} & L_{12} \\ L_{13} & L_{14} \end{bmatrix}=\begin{bmatrix} e^{j\delta/2} & 0 \\ 0 & e^{-j\delta/2} \end{bmatrix} \text{ with}$$

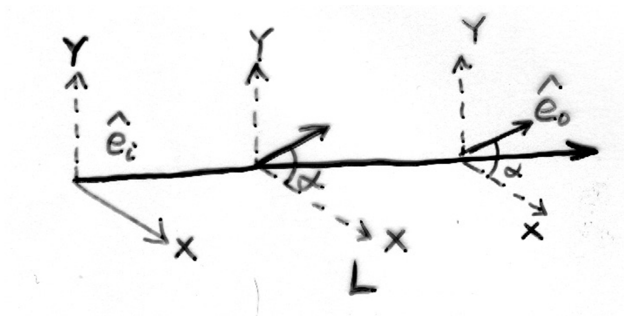
$$\delta=\frac{2\pi dc}{\lambda}\left(\frac{1}{v_X}-\frac{1}{v_Y}\right)=kc\tau_d \text{ and } \tau_d=d\left(\frac{1}{v_X}-\frac{1}{v_Y}\right).$$

For a narrowband light such that $(1/\Delta\nu)\gg\tau_d$, the light transient effect can be neglected.

(iii) Polarization Analyzer Oriented at an Angle α to the X-axis

$$L(\alpha) = \begin{bmatrix} L_{11} & L_{12} \\ L_{13} & L_{14} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}.$$

■ Consider an optical system with the input polarization along X-axis



$$\hat{e}_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\hat{e}_o = \begin{bmatrix} \cos\alpha\cos\alpha \\ \cos\alpha\sin\alpha \end{bmatrix} = L(\alpha)\hat{e}_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \Rightarrow \quad a = \cos^2\alpha, c = \sin\alpha\cos\alpha \ .$$

■ Input polarization is along Y-axis

$$\hat{e}_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\hat{e}_o = \begin{bmatrix} \sin\alpha\cos\alpha \\ \sin\alpha\sin\alpha \end{bmatrix} = L(\alpha)\hat{e}_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

$$\Rightarrow \quad b = \sin\alpha\cos\alpha, \quad d = \sin^2\alpha$$

4.4.1 Coherency Matrix

Consider an EM wave with the state described by

$$U = \begin{bmatrix} u_X(t) \\ u_Y(t) \end{bmatrix}.$$

We define a 2×2 matrix ***J*** to reveal its coherent property

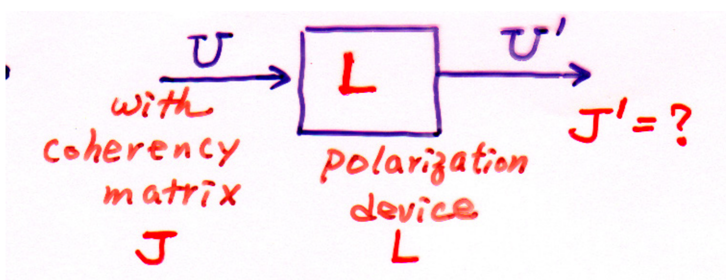
$$J = \langle UU^+ \rangle = \begin{bmatrix} \langle u_X u_X^* \rangle & \langle u_X u_Y^* \rangle \\ \langle u_Y u_X^* \rangle & \langle u_Y u_Y^* \rangle \end{bmatrix} = \begin{bmatrix} J_{XX} & J_{XY} \\ J_{YX} & J_{YY} \end{bmatrix}.$$

Here <…> denotes an infinite time average over *t*.

Note

- $J_{XX}, J_{YY} \geq 0$: because these two parameters denote the average intensity of the *X*- and *Y*-polarization components;
- $J_{XY} = J_{YX}^* \leq \sqrt{J_{XX} J_{YY}}$: A cross-correlation of two polarization components.

We are interested to investigate how the coherency matrix ***J*** of a light field affected by a linear lossless optical device.



To answer the question, first note: $U' = LU$ *and* $U'^+ = U^+L^+$. Therefore,

$$J' \equiv \langle U'U'^+ \rangle = \langle LUU^+L^+ \rangle = L \langle UU^+ \rangle L^+ = LJL^+.$$

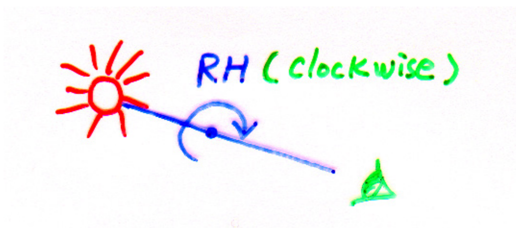
We depict some special cases to illustrate the properties of ***J***

- For a linearly polarized light field in the *X*-direction: $U = \sqrt{\bar{I}}\begin{bmatrix}1\\0\end{bmatrix}$.

$$J \equiv <UU^+> = \bar{I} < \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} > = \bar{I}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

- Linear polarization in the *Y*-direction $U = \sqrt{\bar{I}}\begin{bmatrix}0\\1\end{bmatrix}$:

$$J \equiv <UU^+> = \bar{I}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$



- Linear polarization at 45° to the *X*-direction $U = \sqrt{\bar{I}/2}\begin{bmatrix}1\\1\end{bmatrix}$:

$$J \equiv <UU^+> = \frac{\bar{I}}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}.$$

- Circularly polarized light (LH: +; RH: -) $U = Ae^{-j2\pi\bar{\nu}t}\begin{bmatrix}1\\\pm j\end{bmatrix}$:

$$J_{RH} \equiv <UU^+> = \frac{\bar{I}}{2}\begin{bmatrix}1 & j\\-j & 1\end{bmatrix}.$$

- **Natural Light:**

For natural light, we mean it possesses

(1) equal average intensity in all directions; and

(2) at a fixed direction of polarization that the intensity fluctuates randomly with time.

Therefore, we can express the field of natural light to be

$U = A(t)e^{-j2\pi\bar{v}t}\begin{bmatrix}\cos\theta(t)\\ \sin\theta(t)\end{bmatrix}$, where $\theta(t)$ denotes a slowly varying angle of polarization with respect to the X-axis, which is uniformly distributed on $[-\pi, \pi]$.

$$J = <UU^+> = \bar{I}\begin{bmatrix}<\cos^2\theta(t)> & <\sin\theta(t)\cos\theta(t)> \\ <\sin\theta(t)\cos\theta(t)> & <\sin^2\theta(t)>\end{bmatrix}$$

$$J = <UU^+> = \frac{\bar{I}}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Because

$$<\cos^2\theta(t)> = <\sin^2\theta(t)> = \int_{-\infty}^{\infty}\cos^2\theta(t)dt \overset{Ergodic}{=} \int_{-\pi}^{\pi}\cos^2\theta\, p_\Theta(\theta)d\theta = \frac{1}{2}.$$

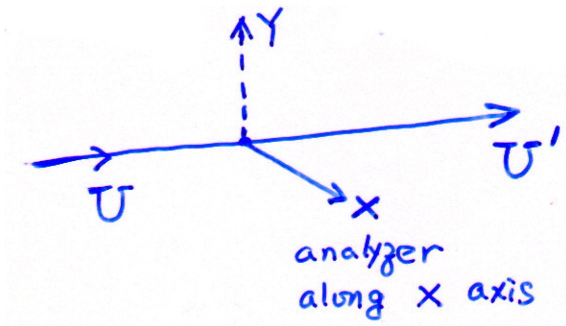
$$<\sin\theta(t)\cos\theta(t)> = 0$$

Note: $J' \equiv <U'U'^+> = LJL^+ = \frac{\bar{I}}{2}LDL^+ = \frac{\bar{I}}{2}LL^+ = \frac{\bar{I}}{2}D = J$. The coherency of natural light cannot be changed by means of a unitary polarization device.

It is worth emphasizing that the coherency matrix of natural light is the same as that of a linearly polarized field, but they do have very different polarization characteristics.

■ Determination of the Components of the Coherency Matrix

(a) Assuming an analyzer to be along the X axis



$$U' = \begin{bmatrix} u_X' \\ u_Y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \end{bmatrix} = \begin{bmatrix} u_X \\ 0 \end{bmatrix}.$$

The transmitted intensity: $\overline{I}_1 = J_{XX}' + J_{YY}' = J_{XX}$.

(b) Analyzer along the *Y*-axis: $\overline{I}_2 = J_{YY}$.

(c) Analyzer at an angle of 45° to the X-axis

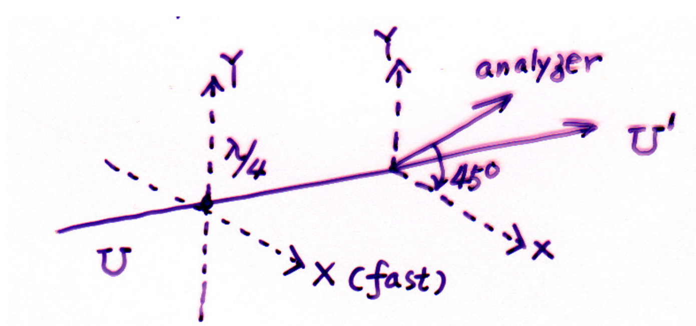
$$U' = \begin{bmatrix} u_X' \\ u_Y' \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u_X \\ u_Y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_X + u_Y \\ u_X + u_Y \end{bmatrix}.$$

$J' = < U'U'^+ >$

The transmitted light intensity

$$J' = < U'U'^+ > = J_{XX}' + J_{YY}' = \frac{1}{2}[J_{XX} + J_{YY}] + \mathrm{Re}[J_{XY}]$$

(d) Combination of a $\lambda/4$ plate and an analyzer



$$U'=\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & j\end{bmatrix}\begin{bmatrix}u_X\\u_Y\end{bmatrix}=\frac{1}{2}\begin{bmatrix}u_X+ju_Y\\u_X+ju_Y\end{bmatrix}$$

$$\therefore \quad J'=<U'U'^{+}>=\begin{bmatrix}<(u_X+ju_Y)(u_X^*-ju_Y^*)> & <(u_X+ju_Y)(u_X^*-ju_Y^*)>\\<(u_X+ju_Y)(u_X^*-ju_Y^*)> & <(u_X+ju_Y)(u_X^*-ju_Y^*)>\end{bmatrix}$$

The transmitted light intensity through the analyzer

$$\overline{I}_2=J_{XX}'+J_{YY}'=\frac{1}{2}[J_{XX}+J_{YY}]+\mathrm{Im}[J_{XY}].$$

Combining the results with the procedures (a), (b), (c), and (d), we can determine J_{XX}, J_{YY} *and* J_{XY} unambiquously.

4.4.2 The Degree of Polarization

Although both of the coherency matrix of natural light and a linearly polarized field are diagonal, they have very different polarization characteristics.

Note

$$J=<UU^{+}>=\begin{bmatrix}<u_Xu_X^*> & <u_Xu_Y^*>\\<u_Yu_X^*> & <u_Yu_X^*>\end{bmatrix}=\begin{bmatrix}J_{XX} & J_{XY}\\J_{YX} & J_{YY}\end{bmatrix},$$

that is Hermitian and nonnegative definite. We can always find a unitary matrix transformation ***U*** (an operation involved lossless rotation and/or phase retardation) such that

$$\begin{aligned}UJU^{+}&=\begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}=\begin{bmatrix}\lambda_2 & 0\\0 & \lambda_2\end{bmatrix}+\begin{bmatrix}\lambda_1-\lambda_2 & 0\\0 & 0\end{bmatrix}\\&=\lambda_2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+(\lambda_1-\lambda_2)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\end{aligned}.$$

Here both λ_1 *and* λ_2 are nonnegative real numbers.

We can define the degree of polarization for a light field as

$$P = \frac{(\lambda_1 - \lambda_2)}{(\lambda_1 + \lambda_2)} = \sqrt{1 - \frac{4\det[J]}{[tr(J)]^2}} = \sqrt{1 - \frac{4\det[J]}{\overline{I}^2}}$$, then

$$\lambda_1 = \frac{1}{2}\overline{I}(1+P), \quad \lambda_2 = \frac{1}{2}\overline{I}(1-P).$$

That implies we can always decompose ***the intensity of a partially polarized wave as the sum of the intensities of two uncorrelated field components***.

4.4.3 First-Order Statistics of the Instantaneous Intensity $I(\vec{r},t)$

After illustrating the physical meaning of a thermal light, it is interesting to raise a question about *what is the probability density function of* $I(\vec{r},t)$ *of a thermal light with the degree of polarization P?*

To answer the question, let us note that

(1) we can always decouple the instantaneous intensity of a partially polarized wave as the sum of two uncorrelated intensity components $I(\vec{r},t) = I_1(\vec{r},t) + I_2(\vec{r},t)$, where $I_1(\vec{r},t)$ and $I_2(\vec{r},t)$ are statistically independent; and

(2) $I_1 \equiv |u_1|^2, I_2 \equiv |u_2|^2$ where both u_1 and u_2 are circularly complex Gaussian fields with

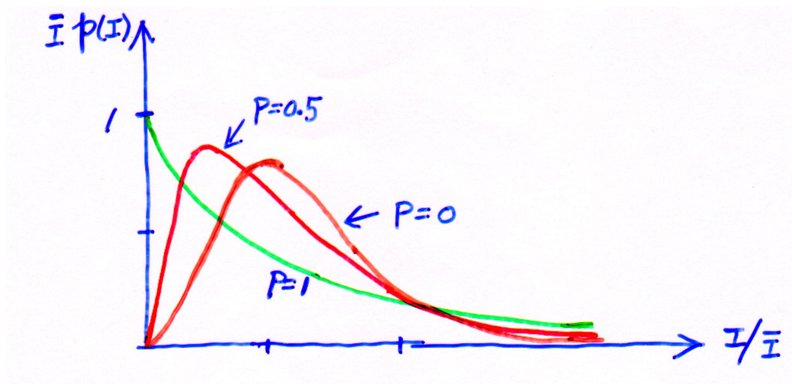
$$p_{I_1}(I_1) = \frac{1}{\overline{I}_1} e^{-I_1/\overline{I}_1} = \frac{2}{(1+P)\overline{I}} e^{-2I_1/[(1+P)\overline{I}]}$$
$$p_{I_2}(I_2) = \frac{1}{\overline{I}_2} e^{-I_2/\overline{I}_2} = \frac{2}{(1-P)\overline{I}} e^{-2I_2/[(1-P)\overline{I}]}.$$

Using characteristic function of a sum of two statistically independent RVs

$$M_I(\omega) = M_{I_1}(\omega)\cdot M_{I_2}(\omega) = \frac{1}{1-j\frac{\omega}{2}(1+P)\overline{I}}\cdot\frac{1}{1-j\frac{\omega}{2}(1-P)\overline{I}}$$

$$= \frac{1}{2P}\left\{\frac{(1+P)}{1-j\frac{\omega}{2}(1+P)\overline{I}} - \frac{(1-P)}{1-j\frac{\omega}{2}(1-P)\overline{I}}\right\}$$

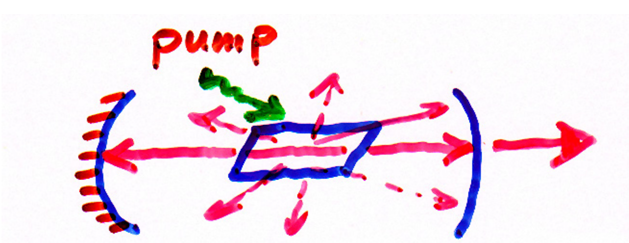
$$\Rightarrow \quad p(I) = FT^{-1}[M_I(\omega)] = \frac{1}{P\overline{I}}\left\{e^{-\frac{2I}{(1+P)\overline{I}}} - e^{-\frac{2I}{(1-P)\overline{I}}}\right\}$$



$$\sigma_I{}^2 = \int_0^\infty (I-\overline{I})^2\, p(I)\, dI = \frac{1+P^2}{2}(\overline{I})^2 .$$

4.5 Laser Light

A laser consists of an active medium excited by an energy source and contained within a resonant cavity that provides feedback. Spontaneous emission from the active medium is reflected from the end mirrors and is reinforced by additional stimulated



emission.

4.5.1 An idealized model of laser with single-mode oscillation

For an optical field from a laser with single-mode oscillation, we can express the light field as $u(t) = S\cos[2\pi\nu_o t - \phi]$, where

$S =$ a known amplitude, and

$\phi =$ unknown phase, which is a RV uniformly distributed on $[-\pi, \pi]$.

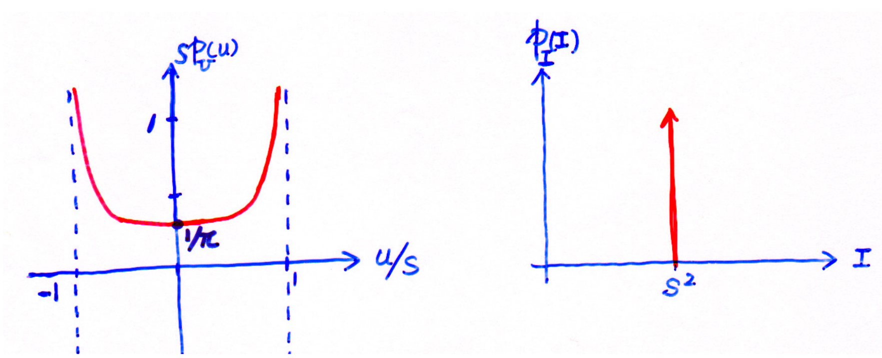
Since ϕ is a stationary random process, we set t=0 and

$$M_U(\omega) = \int_{-\infty}^{+\infty} e^{j\omega u} p_U(u)\, du = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{j\omega S\cos\phi}\, d\phi = J_0(\omega S).$$

$$\Rightarrow \quad p_U(u) = FT^{-1}[J_0(\omega S)] = \begin{cases} \dfrac{1}{\pi\sqrt{S^2 - u^2}}; & |u| \le S \\ 0; & otherwise \end{cases}.$$

Note that $I = |u(t)|^2 = |S\, e^{-j(2\pi\nu_o t - \phi)}|^2 = S^2$, which is a deterministic parameter,

$$\Rightarrow \quad p_I(I) = \delta(I - S^2).$$



4.5.2 Extensions of the Idealized Model

■ Assume the phase undergoes random fluctuations with time, i.e.,

$$u(t) = S\cos[2\pi\nu_o t - \theta(t)] = S\cos\Psi(t) .$$

Define the instantaneous frequency of the laser oscillation as

$$\begin{aligned}\nu(t) &= \frac{1}{2\pi}\frac{d}{dt}\Psi(t) = \frac{1}{2\pi}\frac{d}{dt}[2\pi\nu_o t - \theta(t)] \\ &= \nu_o - \frac{d}{dt}\theta(t) = \nu_o - \nu_R(t)\end{aligned} .$$

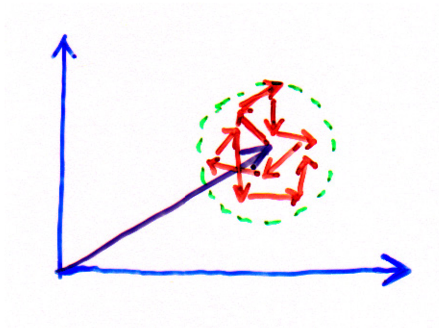
$\nu_R(t)$ is a zero mean, stationary fluctuation, but $\theta(t) = 2\pi\int_{-\infty}^{t}\nu_R(\xi)\,d\xi$ is a nonstationary random process. The fluctuation structure of θ can be calculated from

$$\begin{aligned}D_\theta(t_2,t_1) &= <[\theta(t_2)-\theta(t_1)]^2>_{stat.ave.} \\ &= 4\pi^2 <(\int_{-\infty}^{t_2}\nu_R(\xi)d\xi - \int_{-\infty}^{t_1}\nu_R(\xi)d\xi)^2> \\ &= 4\pi^2 <\{\int_{-\infty}^{\infty} rect[\frac{\xi-(t_1+t_2)/2}{(t_2-t_1)}]\nu_R(\xi)d\xi\}^2>\end{aligned} .$$

Therefore, $D_\theta(t_2,t_1) = 8\pi^2\tau\cdot\int_0^{\tau}(1-\frac{\eta}{\tau})\Gamma_{\nu_R}(\eta)d\eta$. Here $\tau = t_2 - t_1$ and $\Gamma_{\nu_R}(\eta) = <\nu_R(\eta)\nu_R(0)>_{stat.ave.}$.

If the correlation time of ν_R is much shorter than τ (i.e., $\eta/\tau \ll 1, \tau\to\infty$), then

$D_\theta(t_2,t_1) \simeq 8\pi^2\tau\cdot\int_0^{\infty}\Gamma_{\nu_R}(\eta)d\eta \propto \tau$, a characteristic of diffusion process.



■ $u(t) = S\cos[2\pi\nu_o t - \theta(t)] + u_n(t)$, *i.e.*, both of the phase and amplitude are randomly fluctuated in time:

$\theta(t) =$ randomly time-varying phase of the diffusion-type; and

$u_n(t) =$ weakly stationary noise process, which represents a small residual amount of spontaneous emission. Assuming $u_n(t)$ to be a Gaussian random process and independent of $\theta(t)$. Thus,

$u(t) = S + A_n$ =constant amplitude random phasor + circular complex Gaussian phasor.

$$\Rightarrow \quad I = |u(t)|^2 = |S + A_n|^2 \simeq |S|^2 + 2\,\mathrm{Re}\{S^* \cdot A_n\}.$$

Let $\mathbf{S} = S\,e^{j\theta}$ and $\mathbf{A}_n = A_n\,e^{j\phi_n}$. Because A_n, θ, ϕ_n are independent and θ, ϕ_n are uniformly distributed on $[-\pi, \pi]$.

$\Rightarrow \quad 2S^* \cdot A_n$ is a Gaussian RV with zero mean variance

$\sigma_I{}^2 = 4S^2 \cdot \langle A_n{}^2 \rangle \cdot \langle \cos^2(\theta - \phi_n) \rangle = 2S^2 \cdot \langle I_N \rangle = 2I_S \cdot \langle I_N \rangle$.

$$\therefore \quad p_I(I) \simeq \frac{1}{\sqrt{4\pi I_S \langle I_N \rangle}} e^{-(I - I_S)^2 / (4 I_s \langle I_N \rangle)}.$$

This is valid for a CW laser well above the lasing threshold.

■ Multimode Laser Light

Considering a laser operating at the condition of well above threshold (*i.e.*, u_n is much smaller than S_i)

$$u(t) = \sum_{i=1}^{N} S_i \cos[2\pi \nu_i t - \theta_i(t)] .$$

Assuming that these lasing modes oscillate independently

$$\therefore \quad M_U(\omega) = \prod_{i=1}^{N} J_0(\omega S_i) = J_0^{\,N}(\omega\sqrt{\bar{I}/N}) .$$

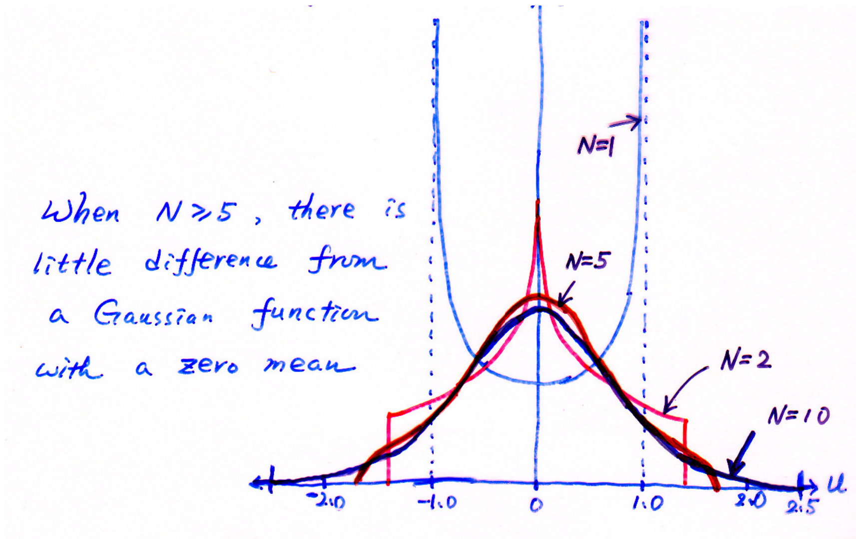
Note: $p_U(u) = FT^{-1}[M_U(\omega)]$,

If N=2 and $S_1 = S_2 = \sqrt{\bar{I}/2}$

$$\Rightarrow \quad p_U(u) = \begin{cases} \dfrac{1}{\pi^2}\sqrt{\dfrac{2}{\bar{I}}}\, K\left(\sqrt{1-\dfrac{u^2}{2\bar{I}}}\right); & |u| < \sqrt{2\bar{I}} \\ 0 & ; \quad otherwise \end{cases} ,$$

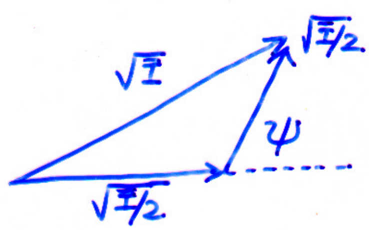
where K denotes the elliptic integral of the first kind.

When $N \geq 5$, there is little difference from a Gaussian function with a zero mean.



Now let us consider the probability density function of the intensity $p_I(I)$.

For simplicity, let $N=2$, we then have $I = \overline{I}/2 + \overline{I}/2 + \overline{I}\cos\Psi = \overline{I}[1+\cos\Psi]$.



Here $\Psi = 2\pi(\nu_1 - \nu_2)t + \theta_1(t) - \theta_2(t)$ is uniformly distributed on $[-\pi, \pi]$.

$$\therefore \quad M_I(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega\overline{I}[1+\cos\Psi]}\, d\Psi = e^{j\omega\overline{I}} J_0(\omega\overline{I}).$$

$$\Rightarrow \quad p_I(I) = FT^{-1}[M_I(\omega)] = \begin{cases} \dfrac{1}{\pi\sqrt{\overline{I}^2 - (I-\overline{I})^2}}; & 0 < I < 2\overline{I} \\ 0 \quad ; & otherwise \end{cases}.$$

As $N \geq 5$, both in field U and intensity *I*, the multimode CW laser light approaches the characteristic of thermal light.