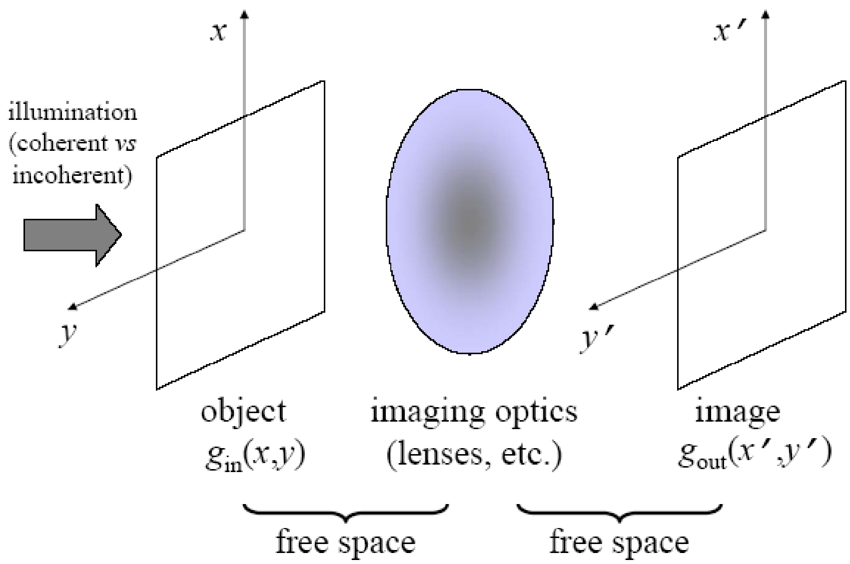
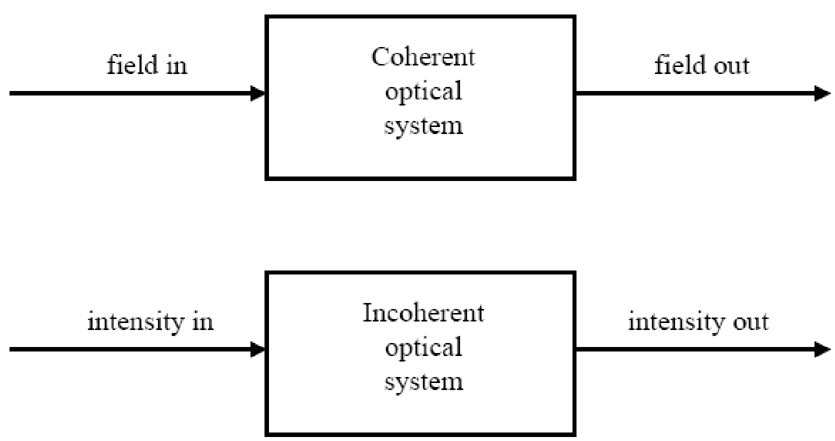
Chapter 1b Overview of image formation

(If you are not familiar with Fourier Optics, try to catch up by reading Goodman's *Introduction to Fourier Optics*)

The imaging problem



Coherent vs incoherent imaging

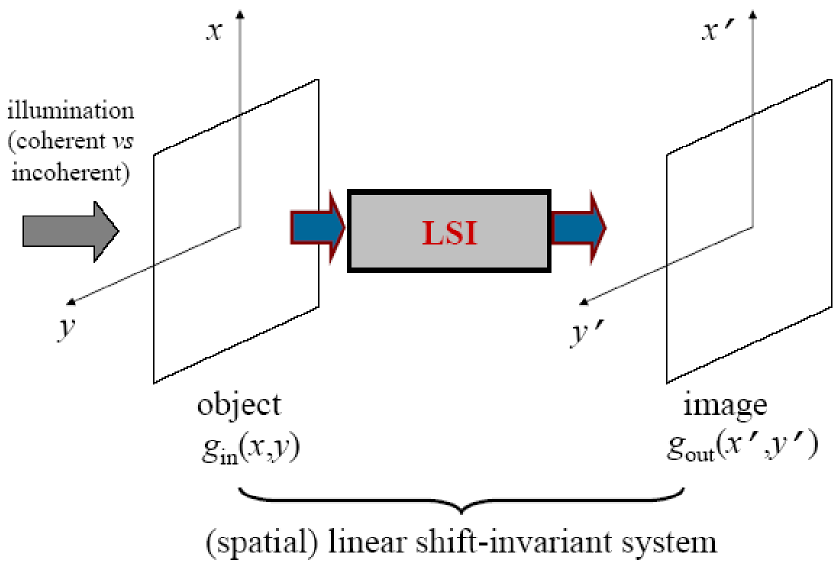


Topics to be discussed in the following Section

→ linear shift invariant (LSI) systems in the representations of real space and spatial frequency

→ mathematical properties of Fourier transform

The imaging problem



Space and spatial frequency representations

SPACE DOMAIN $g(x,y)$

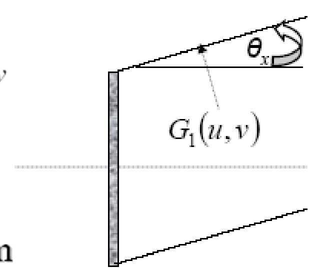
$$G(u,v) = \int g(x,y)\ e^{-i2\pi(ux+vy)}\ dxdy$$

2D Fourier transform

$$g(x,y) = \int G(u,v)\ e^{+i2\pi(ux+vy)}\ dudv$$

2D Fourier integral
aka
inverse 2D Fourier transform

SPATIAL FREQUENCY DOMAIN $G(u,v)$



Monochromatic wave field in the spatial frequency domain

The **2D** Fourier integral

(aka **inverse Fourier transform**)

$$g(x,y) = \int G(u,v)\ e^{+i2\pi(ux+vy)}\ du\,dv$$

superposition

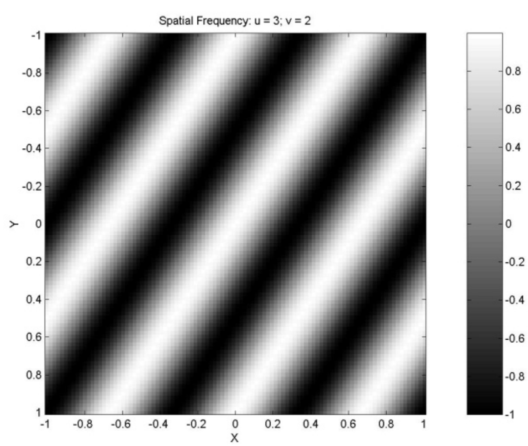
sinusoids

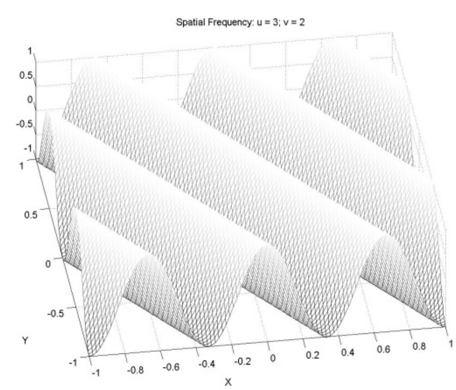
complex weight,
expresses relative amplitude
(magnitude & phase)
of superposed sinusoids

Note: The spatial frequency representation above describes the intersection of a plane wave with the x,y,z=0 plane at t=0 – it is not a plane wave, nor is it a solution of the EM Wave equation.

It is a static 2-D function that changes sinusoidally in a particular direction with a particular cycle length. In the Discrete Fourier Transform, each spatial frequency has an integral number of cycles over the array.

An example of a spatial frequency function is shown in two views below:





Fourier transform properties /1

- Fourier transforms and the delta function

$$\Im\{\delta(x,y)\}=1$$

$$\Im\{\exp[i2\pi(u_0x+v_0y)]\}=\delta(u-u_0)\delta(v-v_0)$$

- Linearity of Fourier transforms

if $\Im\{g_1(x,y)\}=G_1(u,v)$ and $\Im\{g_2(x,y)\}=G_2(u,v)$
then $\Im\{a_1g_1(x,y)+a_2g_2(x,y)\}=a_1G_1(u,v)+a_2G_2(u,v)$
for any pair of complex numbers a_1, a_2.

Fourier transform properties /2

Let $F\{g(x,y)\}=G(u,v)$

- *Shift theorem* (*space* $\rightarrow$ *frequency*)

$$F\{g(x-x_0,y-y_0)\}=G(u,v)\exp[-i2\pi(ux_0+vy_0)]$$

- *Shift theorem* (*frequency* $\rightarrow$ *space*)

$$F\{g(x,y)\exp[i2\pi(u_0x+v_0y)\}=G(u-u_0,v-v_0)$$

- *Scaling theorem*

$$F\{g(ax,by)\}=\frac{1}{|ab|}G(\frac{u}{a},\frac{v}{b})$$

Fourier transform properties /3

Let $\Im\{f(x,y)\}=F(u,v)$ and $\Im\{h(x,y)\}=H(u,v)$

Let $g(x,y)=\int f(x',y')\cdot h(x-x',y-y')\,\mathrm{d}x'\mathrm{d}y'$

- Convolution theorem (space $\rightarrow$ frequency)

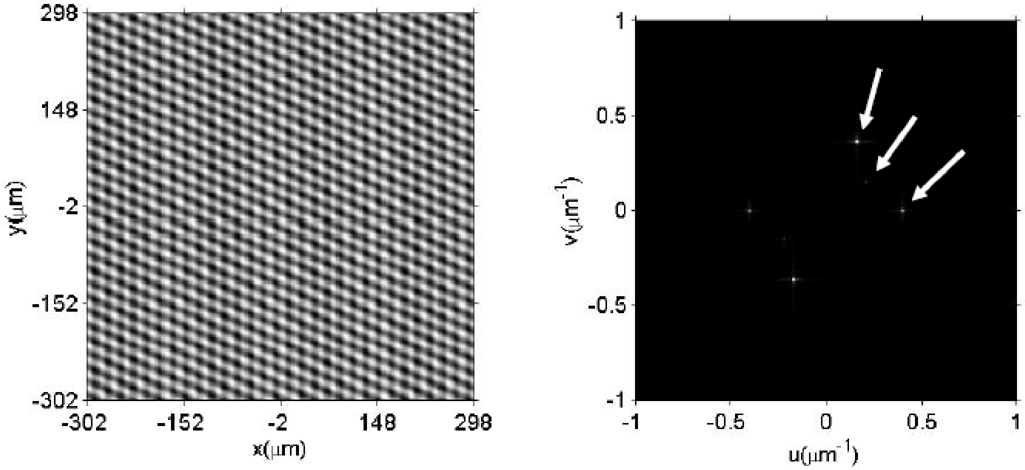
$$\Im\{g(x,y)\}=F(u,v)\cdot H(u,v)$$

Let $Q(u,v)=\int F(u',v')\cdot H(u-u',v-v')\,\mathrm{d}u'\mathrm{d}v'$

- Convolution theorem (frequency $\rightarrow$ space)

$$Q(u,v)=\Im\{f(x,y)\cdot h(x,y)\}$$

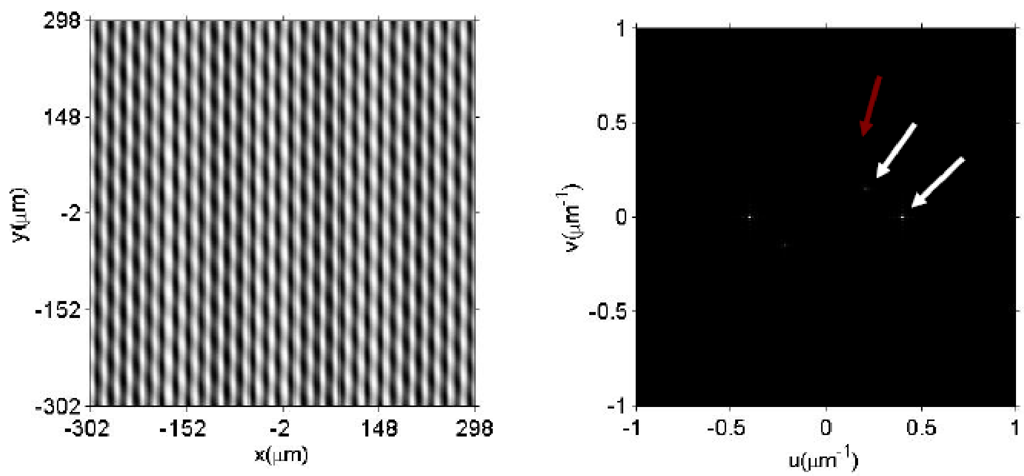
Spatial frequency representation



space domain
3 sinusoids
g(x, y)

Fourier domain
(aka spatial frequency domain)
G(u, v)

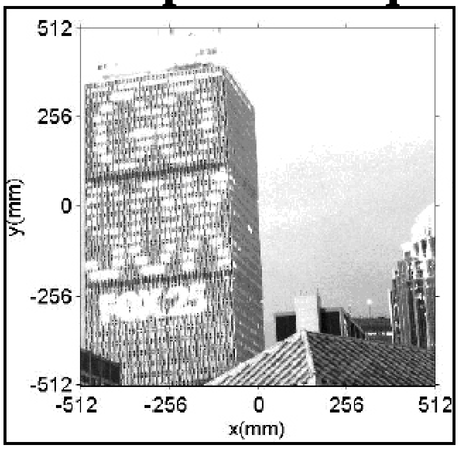
Spatial frequency removal



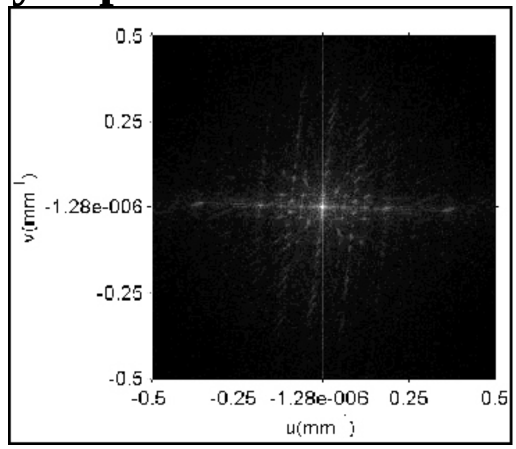
space domain
2 sinusoids (1 removed)

Fourier domain
(aka spatial frequency domain)

Spatial frequency representation

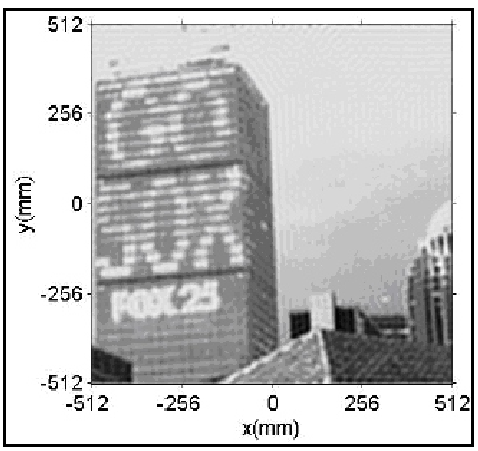


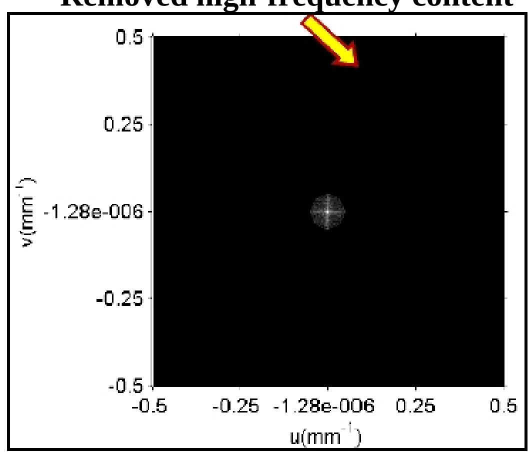
g(x, y)



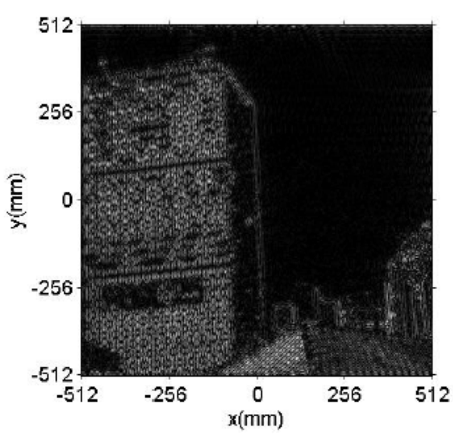
Fourier domain:
G(u, v)=F{g(x, y)}

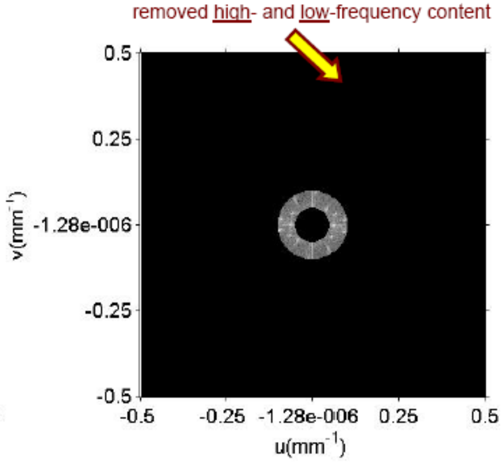
Low-pass filtering



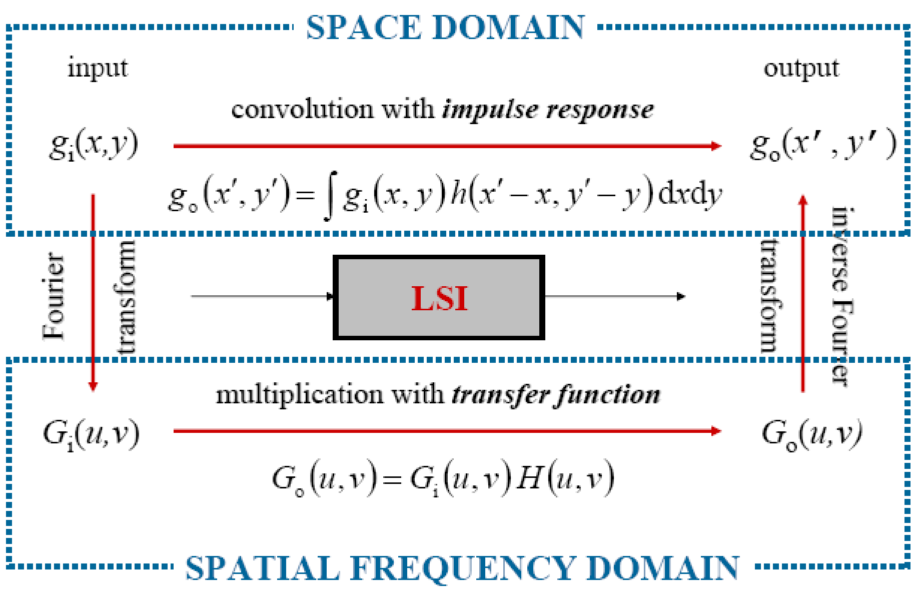


Band-pass filtering





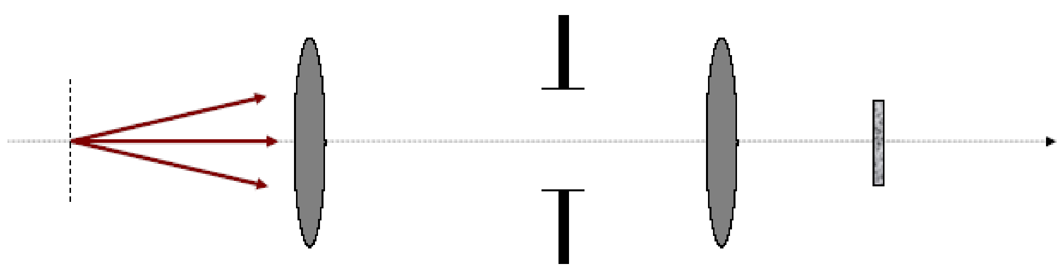
2D linear shift invariant systems



- **Coherent image formation**

– **space domain description**: impulse response
– **spatial frequency domain description**: coherent transfer function

Impulse response & transfer function

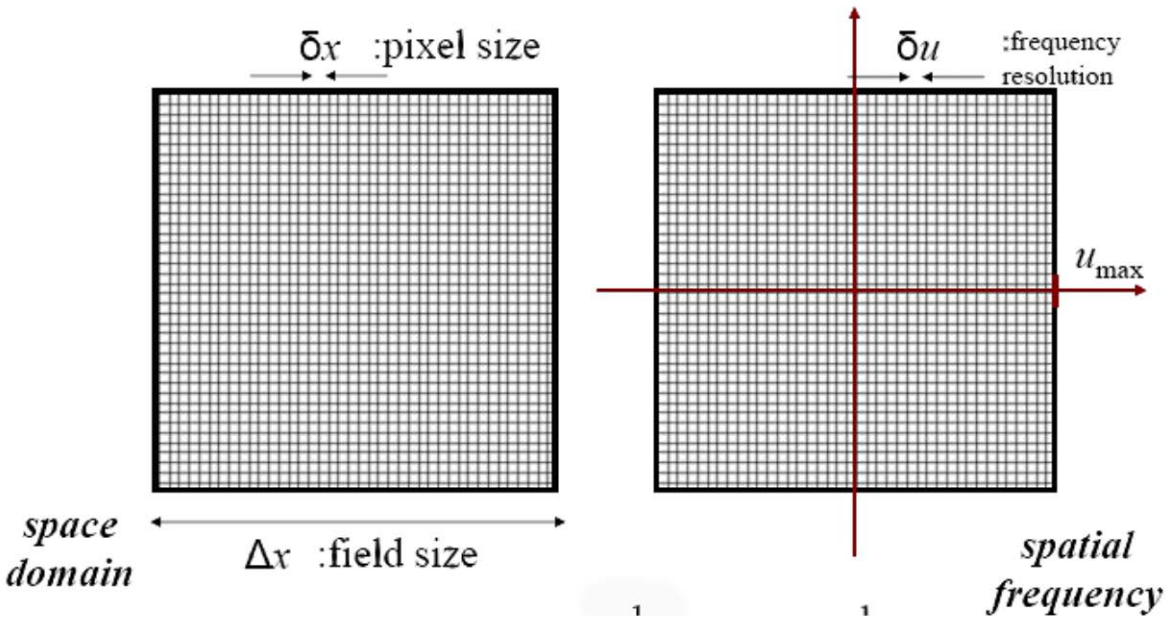


A point source at the input plane ...

... results not in a point image but in a diffraction pattern $h(x',y')$

1. Point source at the origin ↔ delta function $\delta(x, y)$
2. $h(x', y')$ is the impulse response of the system. More commonly, $h(x',y')$ is called the **Coherent Point Spread Function (Coherent PSF).**

Sampling space *and* frequency



$$\delta x = \frac{1}{2u_{max}} \qquad \delta u = \frac{1}{\Delta x} = \frac{1}{N\delta x}$$

The Space–Bandwidth Product

Nyquist relationships:

from space → spatial frequency domain: $$u_{max} = \frac{1}{2\delta x}$$

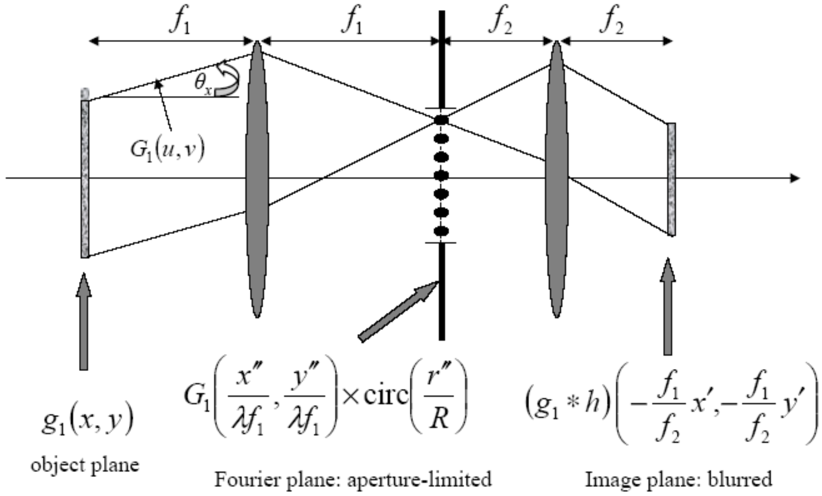
from spatial frequency → space domain: $$\frac{\Delta x}{2} = \frac{1}{2\delta u}$$

$$\frac{\Delta x}{\delta x} = \frac{2u_{max}}{\delta u} \equiv N$$: 1D Space–Bandwidth Product (SBP)

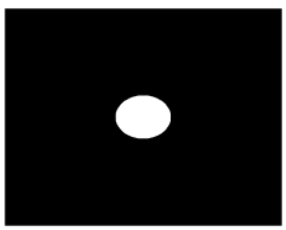
aka number of pixels in the space domain

2D SBP ~ N^2

4f optical imaging system with an aperture on the focal plane

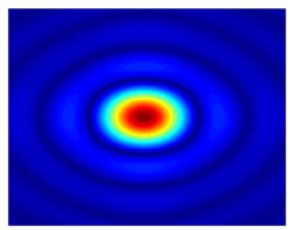


In this case, image is blurred by the low-pass filtering effect of a finite aperture locating on the focal-plane.



Transfer function:
circular aperture

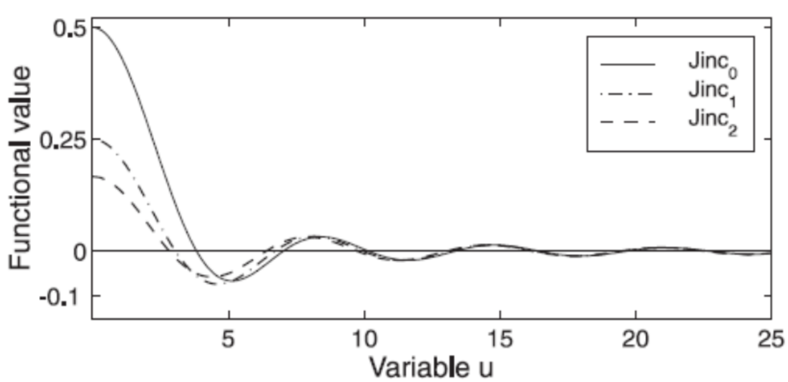
$$\mathrm{circ}\left(\frac{r''}{R}\right)$$



Impulse response:
Airy function

$$\mathrm{jinc}\left(\frac{r'R}{\lambda f_2}\right)$$

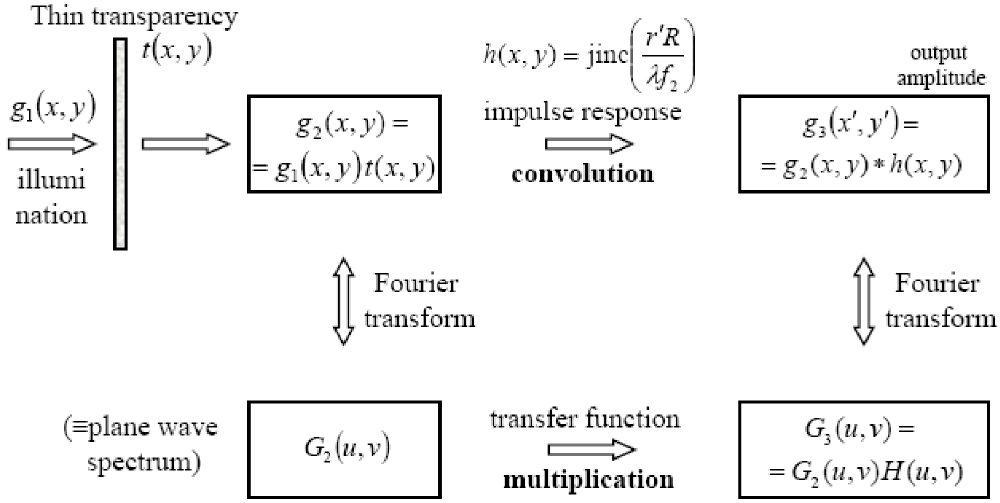
$$Jinc_n(u) = \frac{1}{u^{2n+2}}\int_0^u v^{2n+1}J_0(v)dv, \quad Jinc(u) = \frac{J_1(u)}{u}.$$



Note: circ=@(ri) (abs(ri)<=1.)

Coherent imaging
as a linear, shift-invariant system

Example: 4F system with *circular* aperture @ Fourier plane



$$H(u,v) = \mathrm{circ}\left(\frac{\lambda f\sqrt{u^2+v^2}}{R}\right)$$

The transfer function of an 4*f* imaging system with a rectangular aperture in the Fourier plane:

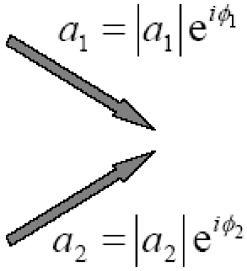
$$H(u,v) = \mathrm{rect}\left(\frac{\lambda f u}{a}\right)\mathrm{rect}\left(\frac{\lambda f v}{b}\right)$$

The image formed with an aperture-limited spatial filtering can be depicted with

$$g_{image}(x',y') = g_3\left(-\frac{f_1}{f_2}x', -\frac{f_1}{f_2}y'\right) = g_2 * h \xrightarrow{h\to\delta} g_2.$$

Imaging with incoherent light

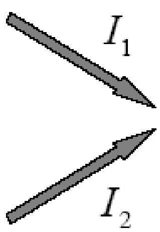
Coherent vs incoherent beams



Mutually coherent: superposition field *amplitude* is described by *sum of complex amplitudes*

$$a = a_1 + a_2 = |a_1| e^{i\phi_1} + |a_2| e^{i\phi_2}$$

$$I = |a|^2 = |a_1 + a_2|^2$$

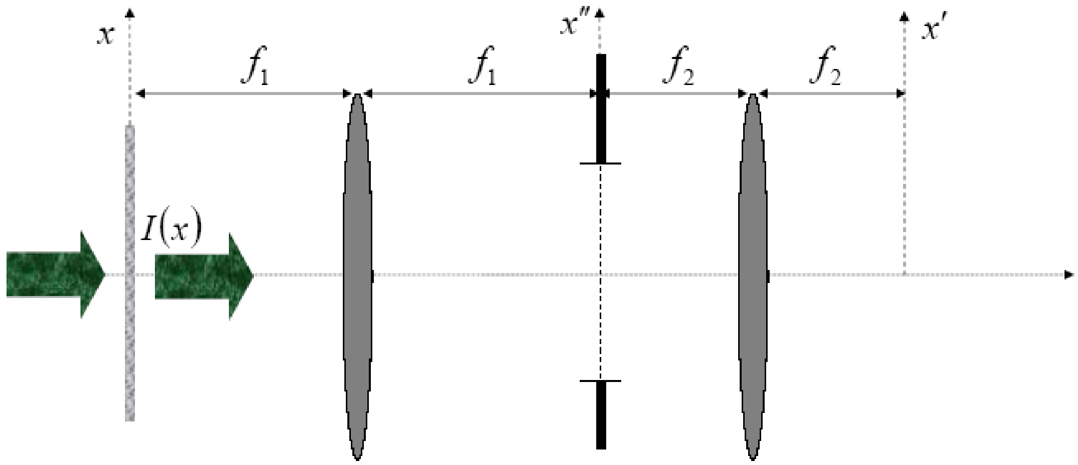


Mutually incoherent: superposition field *intensity* is described by *sum of intensities*

$$I = I_1 + I_2$$

(the phases of the individual beams vary randomly with respect to each other; hence, we would need statistical formulation to describe them properly — *statistical optics*)

Imaging with spatially incoherent light

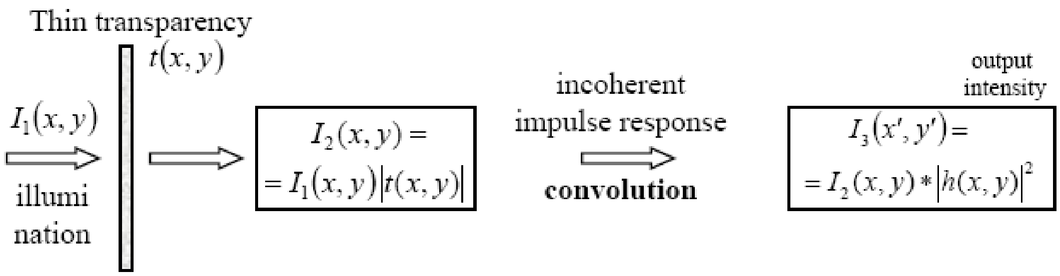


Generalizing:
thin transparency with
sp. incoherent illumination

$$I(x') = \int I(x)\left|h(x'-x)\right|^2 \, \mathrm{d}x$$

intensity at the output
of the imaging system

Incoherent imaging
as a linear, shift-invariant system

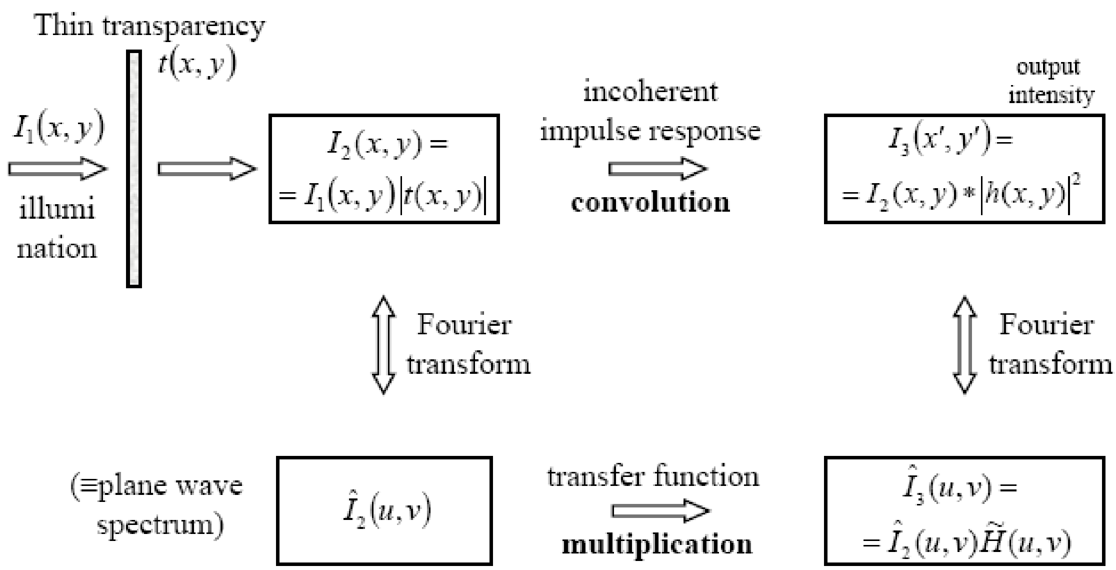


Incoherent imaging is ***linear in intensity***
with incoherent impulse response (iPSF)

$$\tilde{h}(x,y) = \left|h(x,y)\right|^2$$

where $h(x,y)$ is the coherent impulse response (cPSF)

Incoherent imaging
as a linear, shift-invariant system



transfer function of incoherent system: $\tilde{H}(s_x, s_y)$ **optical transfer function (OTF)**

some terminology ...

$H(u, v)$ Amplitude transfer function (coherent)

$\tilde{H}(u, v)$ Optical Transfer Function (OTF) (incoherent)

$|\tilde{H}(u, v)|$ Modulation Transfer Function (MTF)

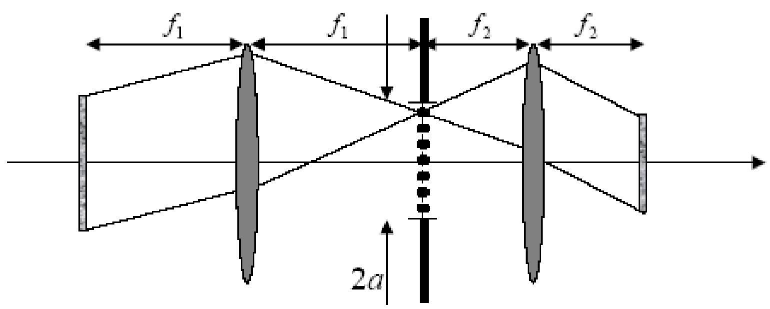
Coherent vs incoherent imaging

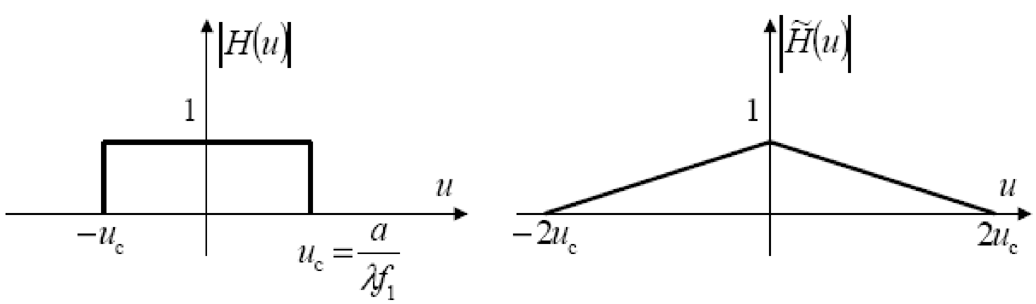
Coherent impulse response (field in ⇒ field out)	$h(x,y)$
Coherent transfer function (FT of field in ⇒ FT of field out)	$H(u,v)=\mathrm{FT}\{h(x,y)\}$
Incoherent impulse response (intensity in ⇒ intensity out)	$\tilde{h}(x,y)=\lvert h(x,y)\rvert^2$
Incoherent transfer function (FT of intensity in ⇒ FT of intensity out)	$\tilde{H}(u,v)=\mathrm{FT}\{\tilde{h}(x,y)\}$ $=H(u,v)\otimes H(u,v)$

$\lvert\tilde{H}(u,v)\rvert$: Modulation Transfer Function (MTF)

$\tilde{H}(u,v)$: Optical Transfer Function (OTF)

Coherent vs incoherent imaging

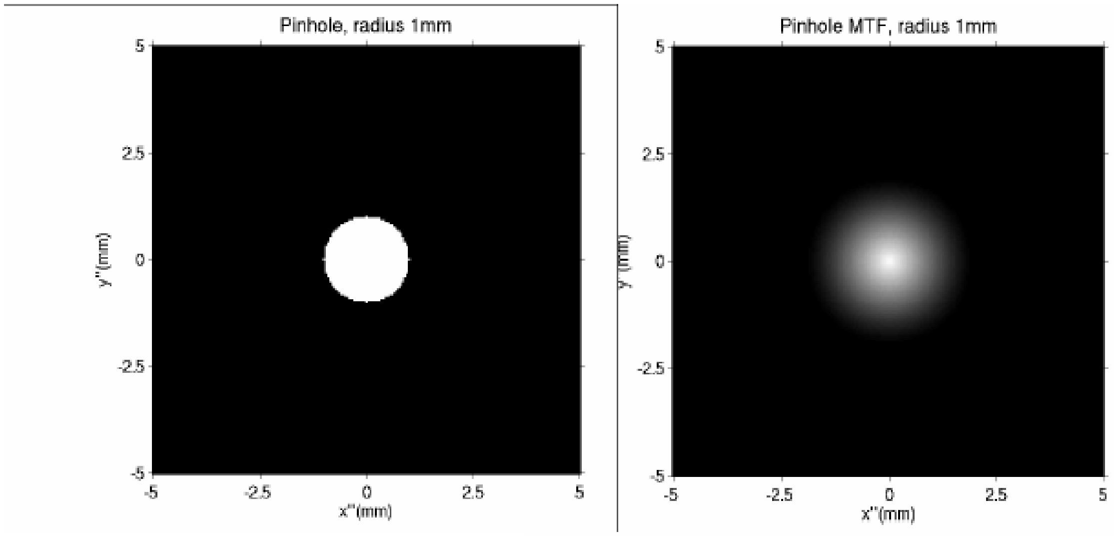




Coherent illumination

Incoherent illumination

MTF of circular aperture

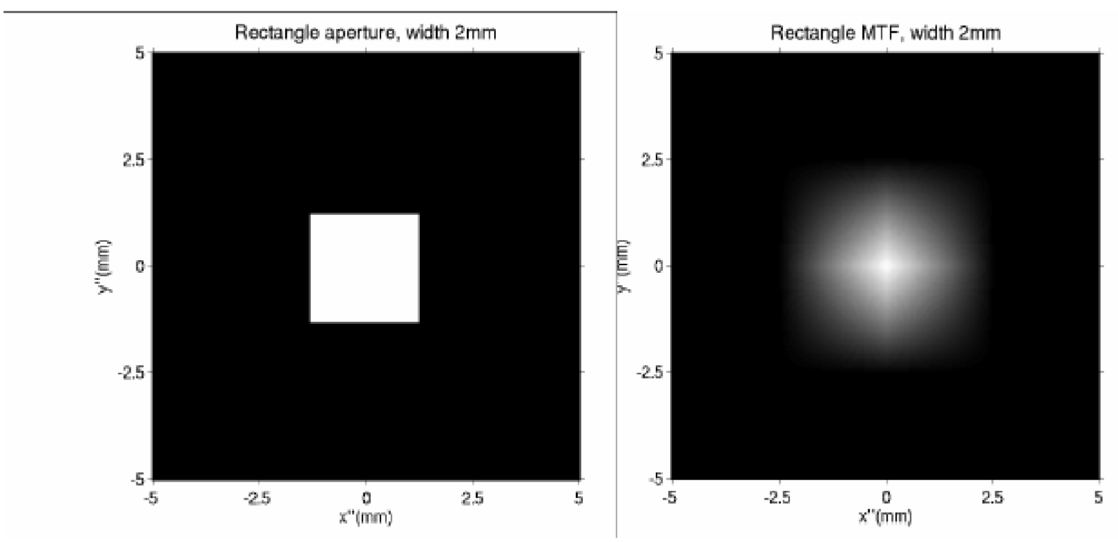


f_1=20cm
λ=0.5μm

physical aperture

filter shape (MTF)

MTF of rectangular aperture



f_1=20cm
λ=0.5μm

physical aperture

filter shape (MTF)

Imaging with polychromatic light

Monochromatic, spatially incoherent response
at wavelength λ_0:

$$I(x', y'; \lambda_0) = \iint I(x, y; \lambda_0) \left| h(x' - x, y' - y; \lambda_0) \right|^2 \, \mathrm{d}x\mathrm{d}y$$

Polychromatic (temporally <u>and</u> spatially incoherent)
response:

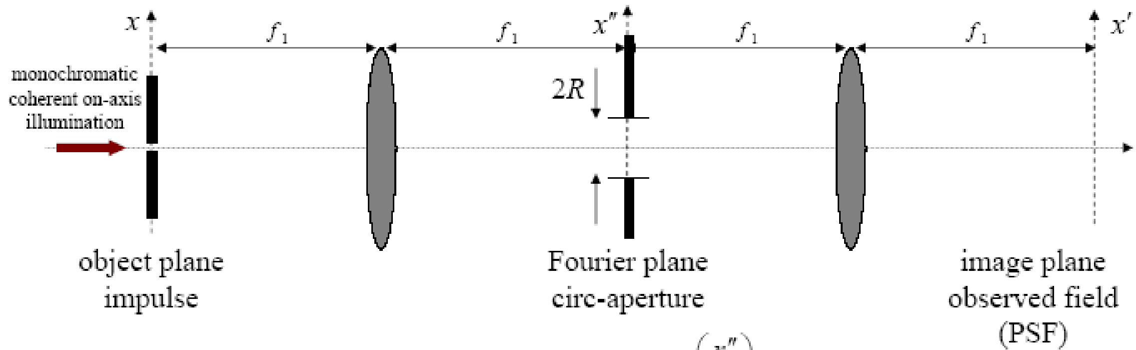
$$\begin{aligned} I(x', y') &= \int I(x', y'; \lambda_0) \, \mathrm{d}\lambda_0 \\ &= \int \iint I(x, y; \lambda_0) \left| h(x' - x, y' - y; \lambda_0) \right|^2 \, \mathrm{d}x\mathrm{d}y \, \mathrm{d}\lambda_0 \end{aligned}$$

Comments on coherent *vs* incoherent

- Incoherent generally gives better image quality:
  - no ringing artifacts
  - no speckle
  - higher bandwidth (even though higher frequencies are attenuated because of the MTF roll-off)
- However, incoherent imaging is insensitive to phase objects
- Polychromatic imaging introduces further blurring due to chromatic aberration (dependence of the MTF on wavelength)

See: cohImaging.m and incohImaging.m

Connection between PSF and NA



$$g_{\text{in}}(x,y)=\delta(x)\delta(y)$$

$$H(x'',y'')=\text{circ}\left(\frac{r''}{R}\right)$$

$\mathfrak{F}$ Fourier transform

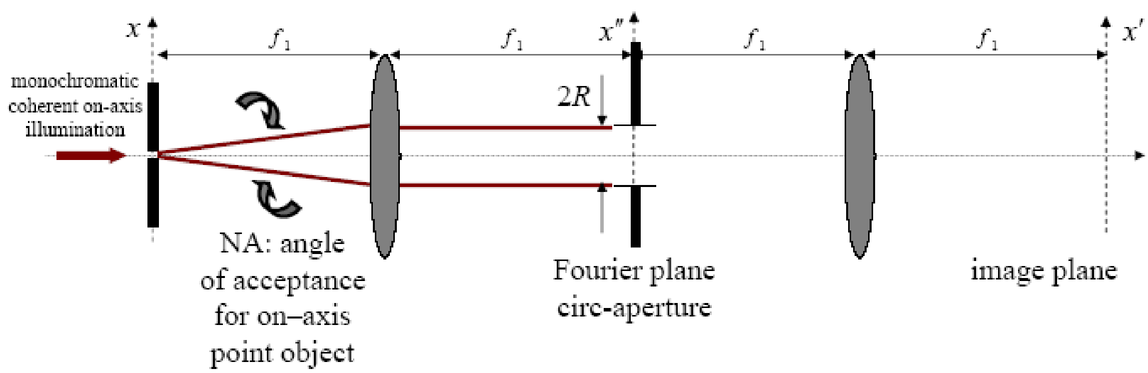
$$\text{jinc}(.,.)\equiv 2\frac{J_1\left(2\pi\frac{R}{f_1}\frac{r'}{\lambda}\right)}{2\pi\frac{R}{f_1}\frac{r'}{\lambda}}$$

(unit magnification)

$r''=\sqrt{x''^2+y''^2}$ radial coordinate @ Fourier plane

$r'=\sqrt{x'^2+y'^2}$ radial coordinate @ image plane

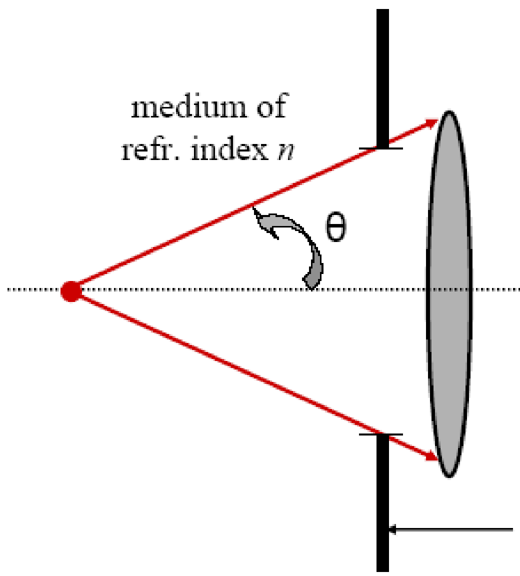
Connection between PSF and NA



$$\text{jinc}\left(-2\frac{R}{f_1}\frac{x'}{\lambda},-2\frac{R}{f_1}\frac{y'}{\lambda}\right)\equiv 2\frac{J_1\left(2\pi\frac{R}{f_1}\frac{r'}{\lambda}\right)}{2\pi\frac{R}{f_1}\frac{r'}{\lambda}}=2\frac{J_1\left(2\pi(\text{NA})\frac{r'}{\lambda}\right)}{2\pi(\text{NA})\frac{r'}{\lambda}}$$

Numerical Aperture (NA) by definition: $(\text{NA})\equiv\frac{R}{f_1}$

Numerical Aperture and Speed (or F–Number)



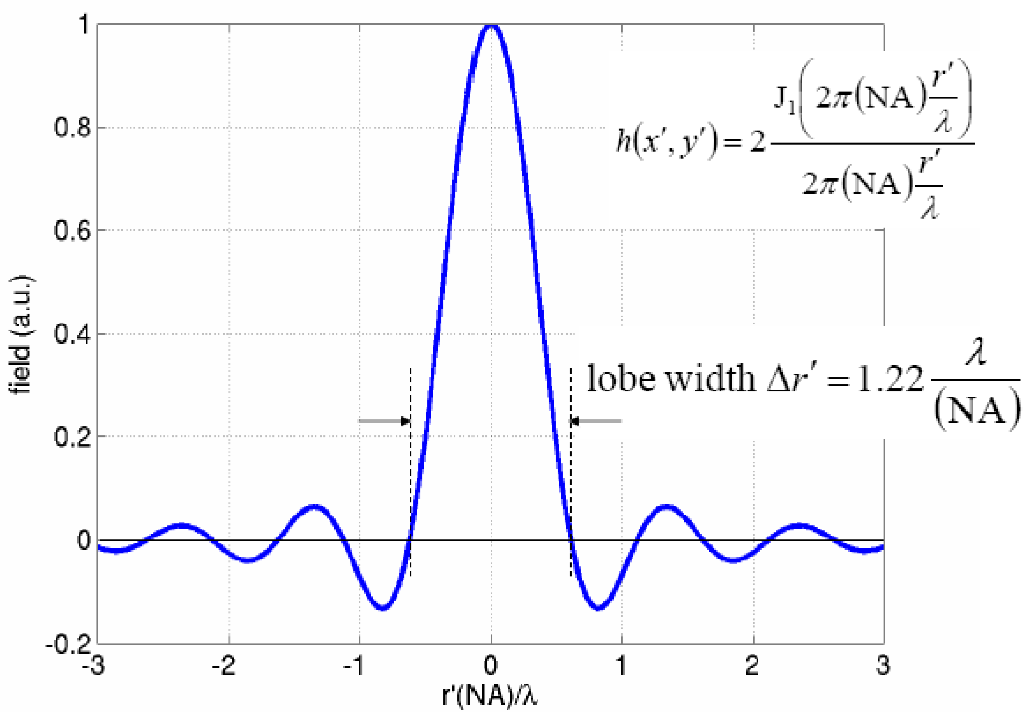
θ: half-angle subtended by the imaging system from an *axial* object

Numerical Aperture
$(\mathrm{NA}) = n \sin\theta$

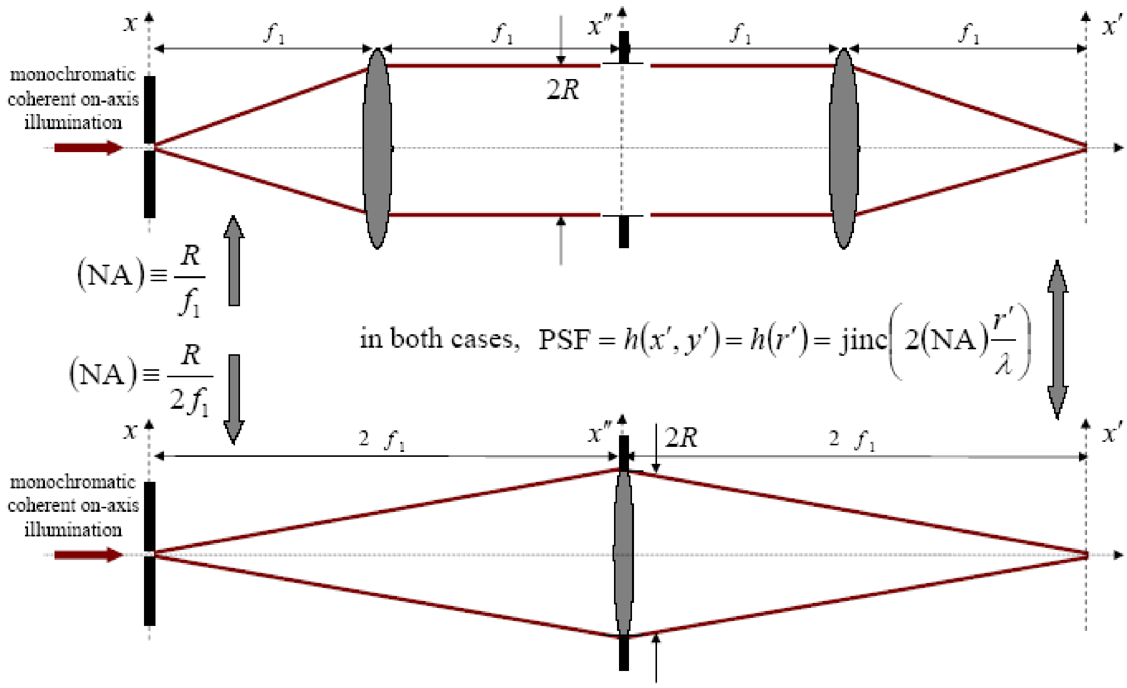
Speed (f/#)=1/2(NA)
pronounced f-number, e.g.
f/8 means (f/#)=8.

Aperture stop
the physical element which limits the angle of acceptance of the imaging system

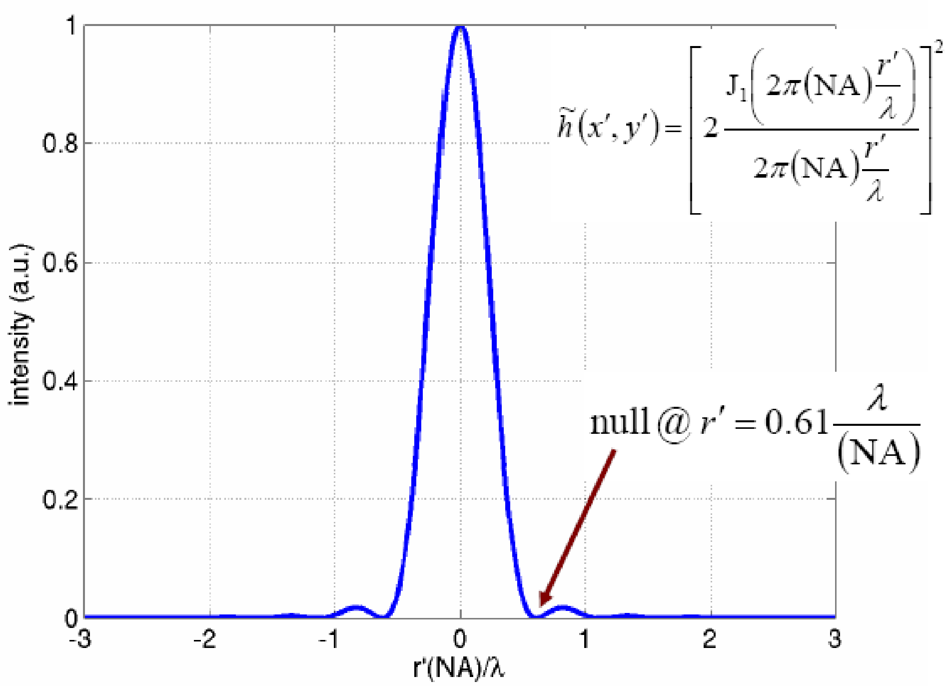
Connection between PSF and NA



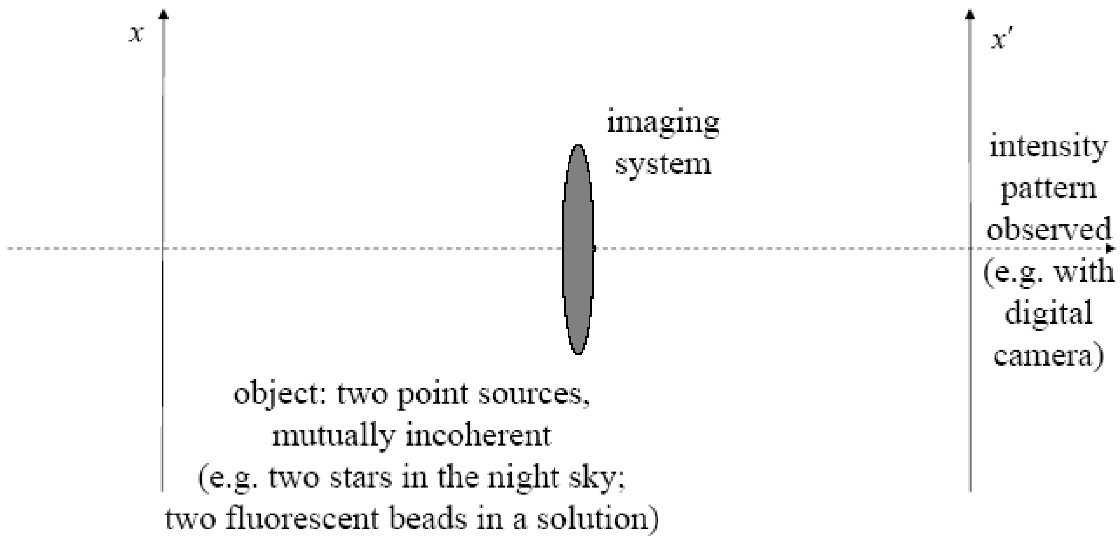
NA in unit–mag imaging systems



The incoherent case: $\tilde{h}(x', y') = |h(x', y')|^2$

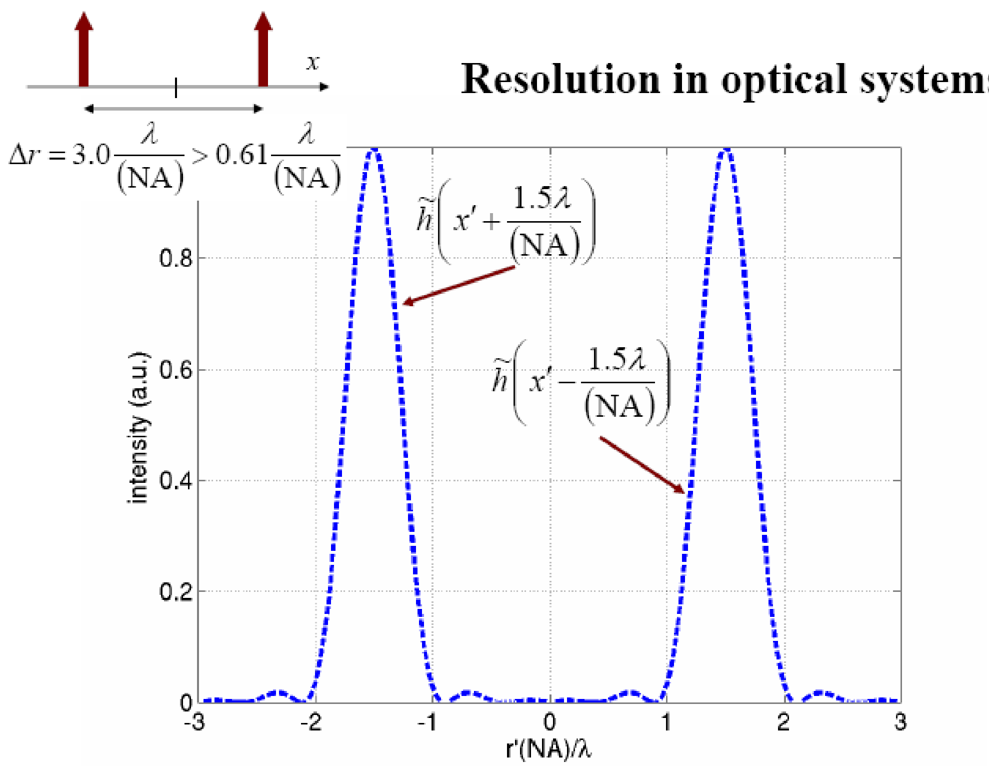


The two–point resolution problem

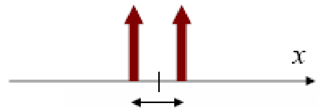


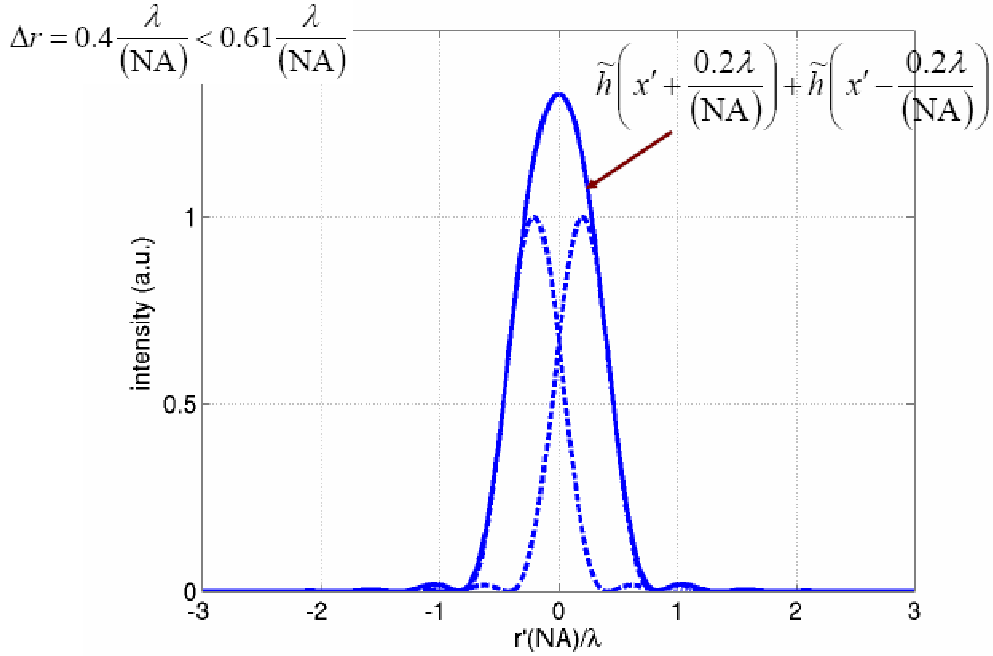
The resolution question [Rayleigh, 1879]: when do we cease to be able to resolve the two point sources (*i.e.*, tell them apart) due to the blurring introduced in the image by the finite (NA)?

Resolution in optical systems

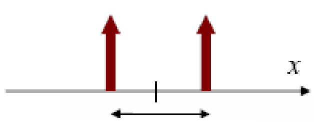


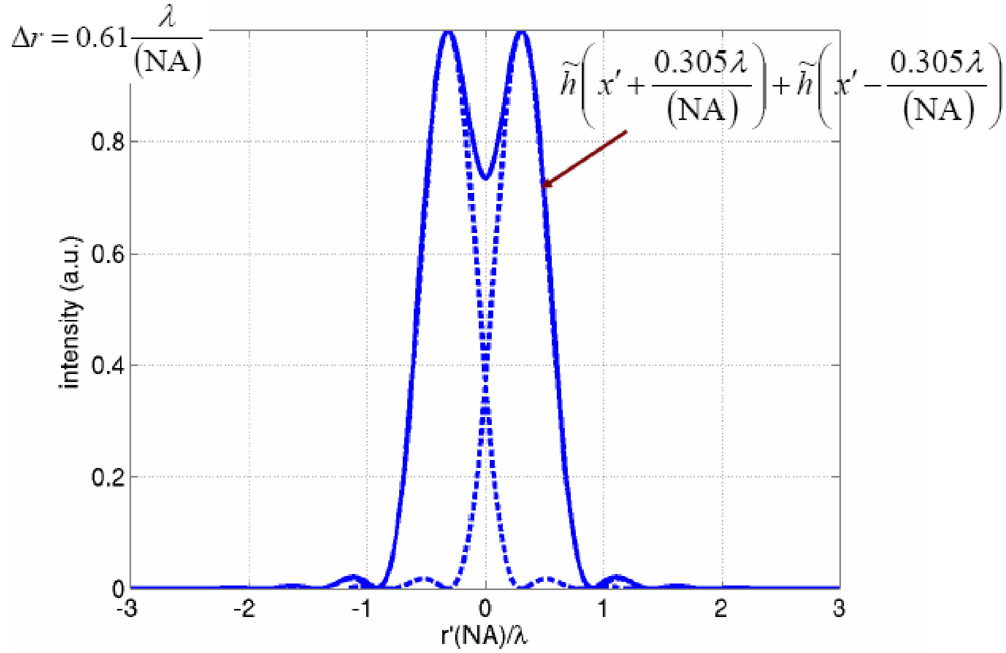
Resolution in optical systems



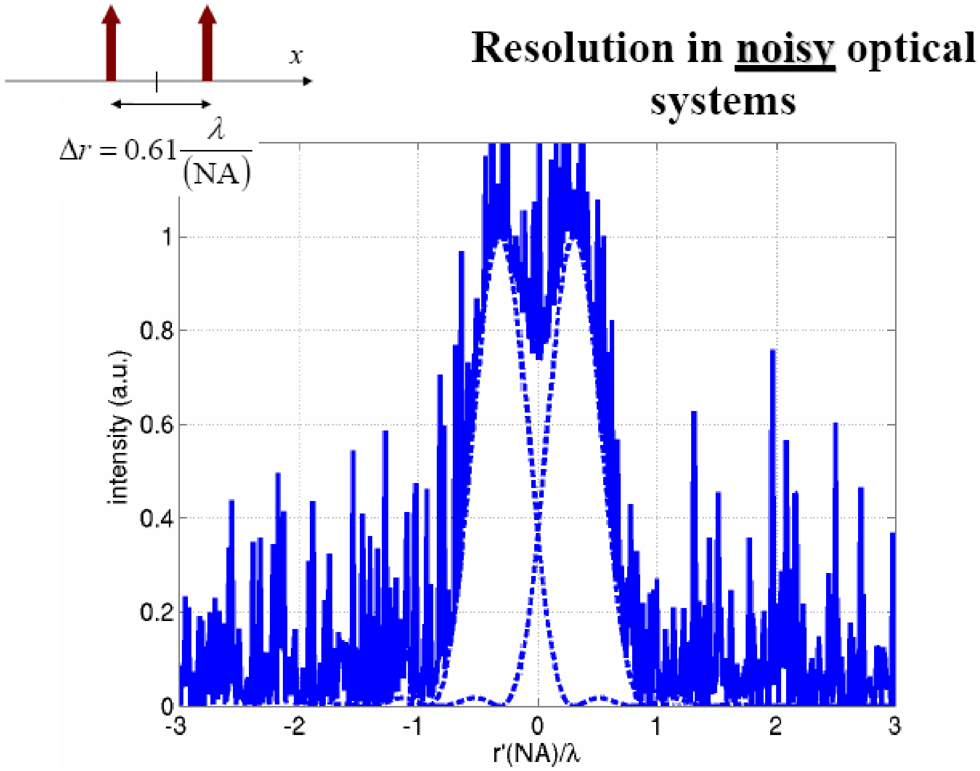


Resolution in optical systems

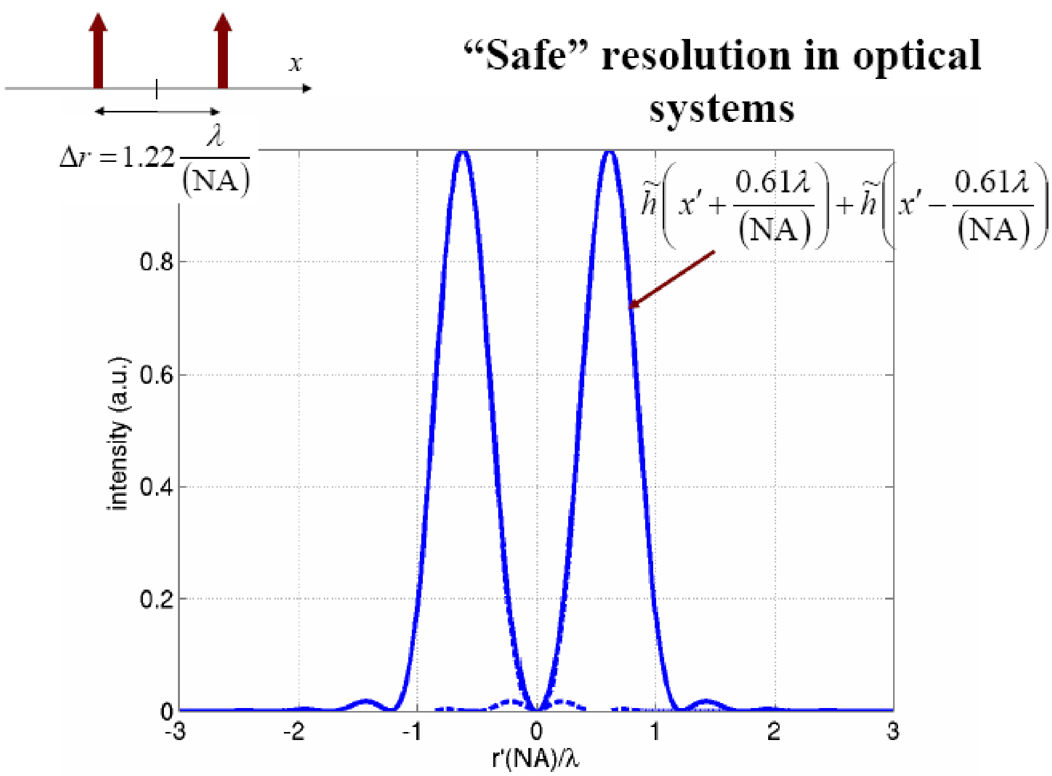




Resolution in **noisy** optical systems



"Safe" resolution in optical systems



Diffraction–limited resolution (safe)

Two point objects are "**<u>just resolvable</u>**" (limited by diffraction only)
if they are separated by:

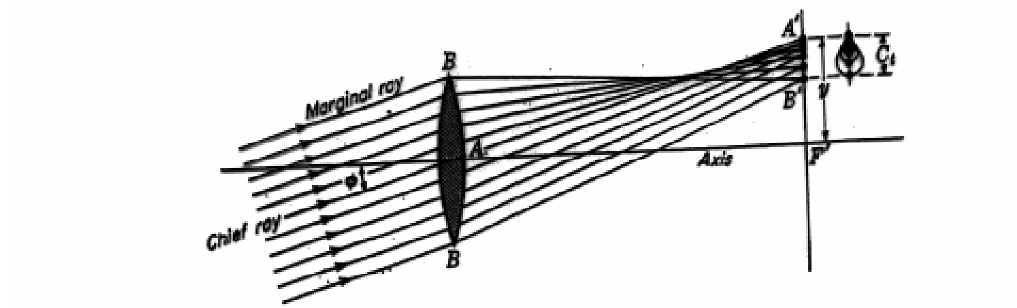
Two–dimensional systems (rotationally symmetric PSF)	One–dimensional systems (e.g. slit–like aperture)
Safe definition: (one–lobe spacing) $\Delta r' = 1.22 \frac{\lambda}{(\text{NA})}$	$\Delta x' = \frac{\lambda}{(\text{NA})}$
Pushy definition: (1/2–lobe spacing) $\Delta r' = 0.61 \frac{\lambda}{(\text{NA})}$	$\Delta x' = 0.5 \frac{\lambda}{(\text{NA})}$

You will see different authors giving different definitions.
Rayleigh in his original paper (1879) noted the issue of noise
and warned that the definition of "just–resolvable" points
is system– or application –dependent

Also affecting resolution: aberrations

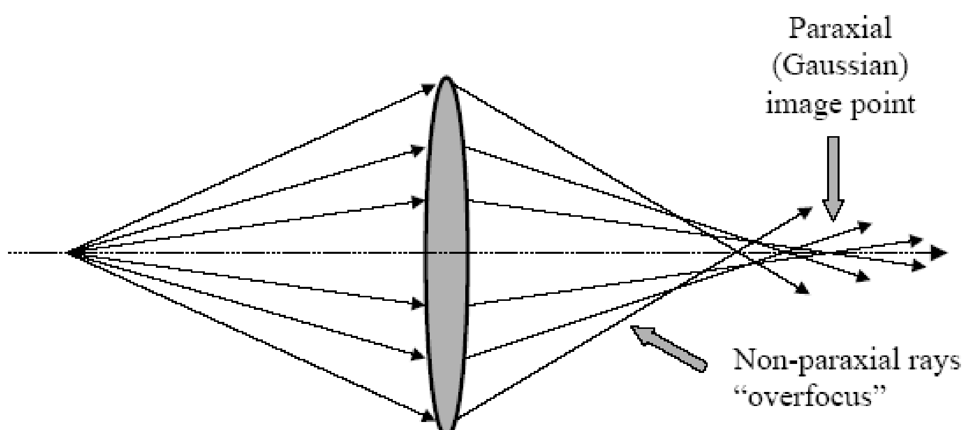
All our calculations have assumed “geometrically perfect” systems, i.e. we calculated the wave–optics behavior of systems which, in the paraxial geometrical optics approximation would have imaged a point object onto a perfect point image.

The effect of aberrations (calculated with non–paraxial geometrical optics) is to blur the “geometrically perfect” image; including the effects of diffraction causes additional blur.



geometrical optics picture

Aberrations: geometrical



Spherical aberration

- Origin of aberrations: nonlinearity of Snell’s law ($n\,\sin\theta$=const., whereas linear relationship would have been $n\theta$=const.)
- Aberrations cause practical systems to perform *worse* than diffraction-limited
- Aberrations are best dealt with using optical design software (Code V, Oslo, Zemax); optimized systems usually resolve ~3-5λ (~1.5-2.5μm in the visible)

Astigmatism

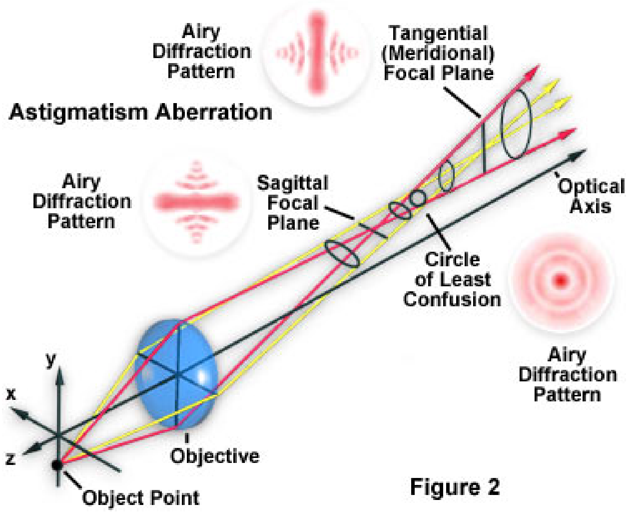


Figure 2

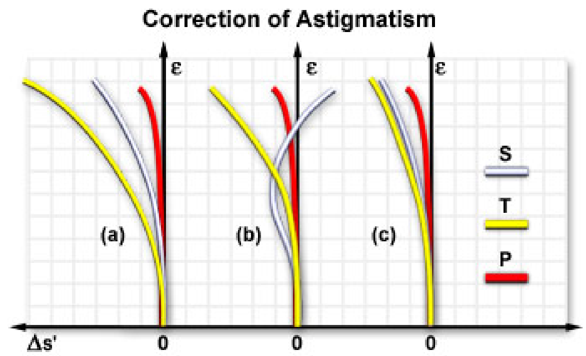


Figure 3

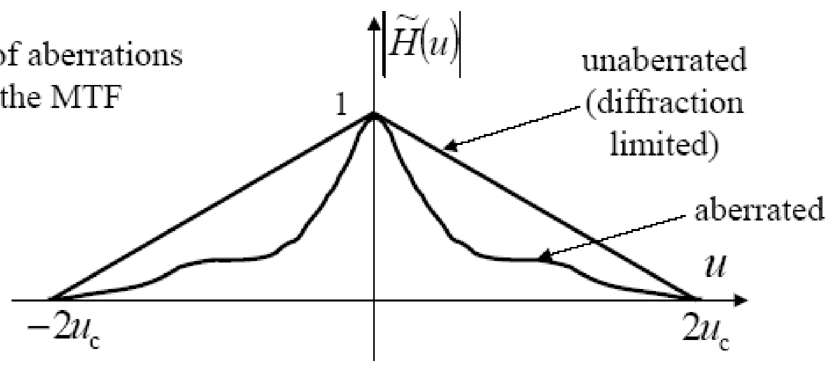
Aberrations: wave

Aberration-free impulse response $h_{\substack{\text{diffraction}\\\text{limited}}}(x, y)$

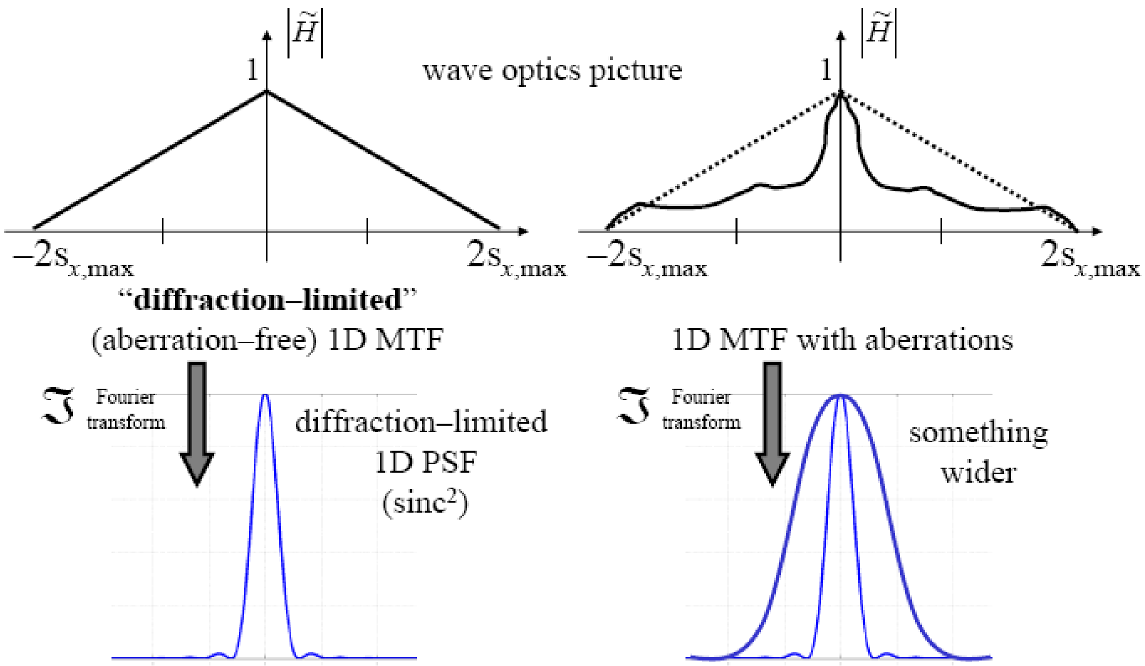
Aberrations introduce additional phase delay to the impulse response

$$h_{\text{aberrated}}(x, y) = h_{\substack{\text{diffraction}\\\text{limited}}}(x, y)\, e^{i\varphi_{\text{aberration}}(x,y)}$$

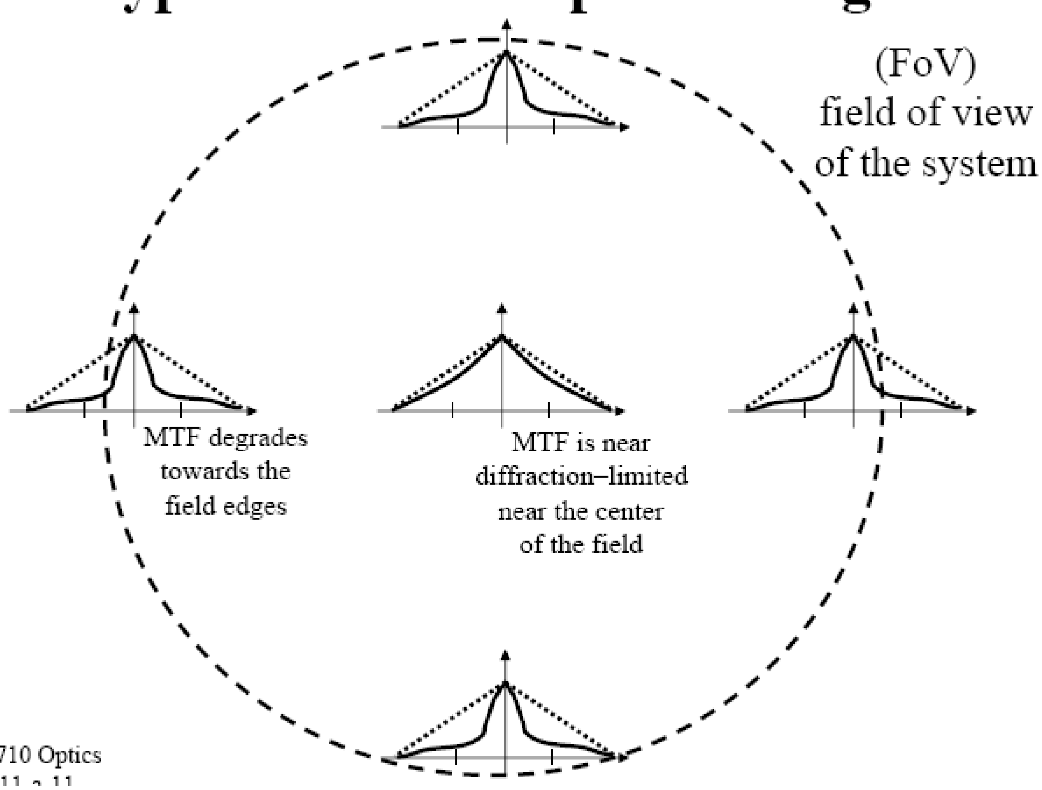
Effect of aberrations on the MTF



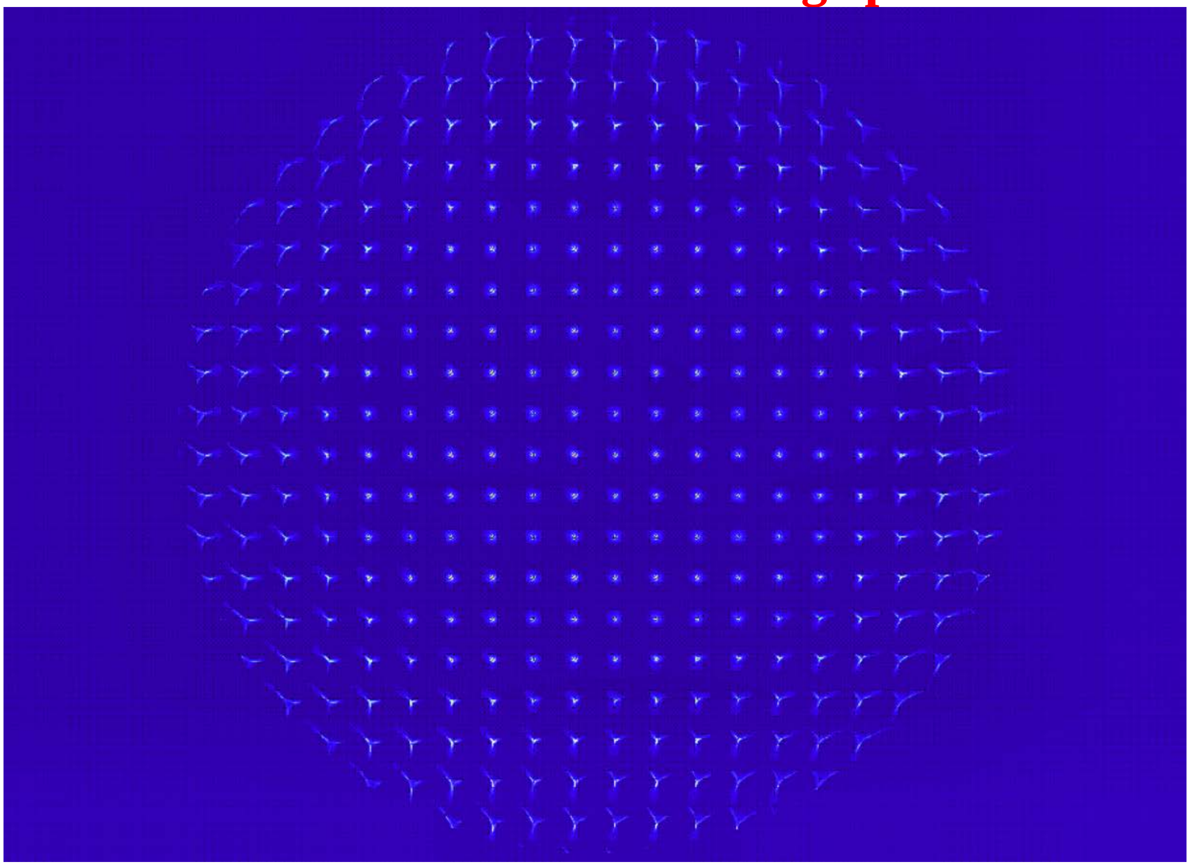
Also affecting resolution: aberrations



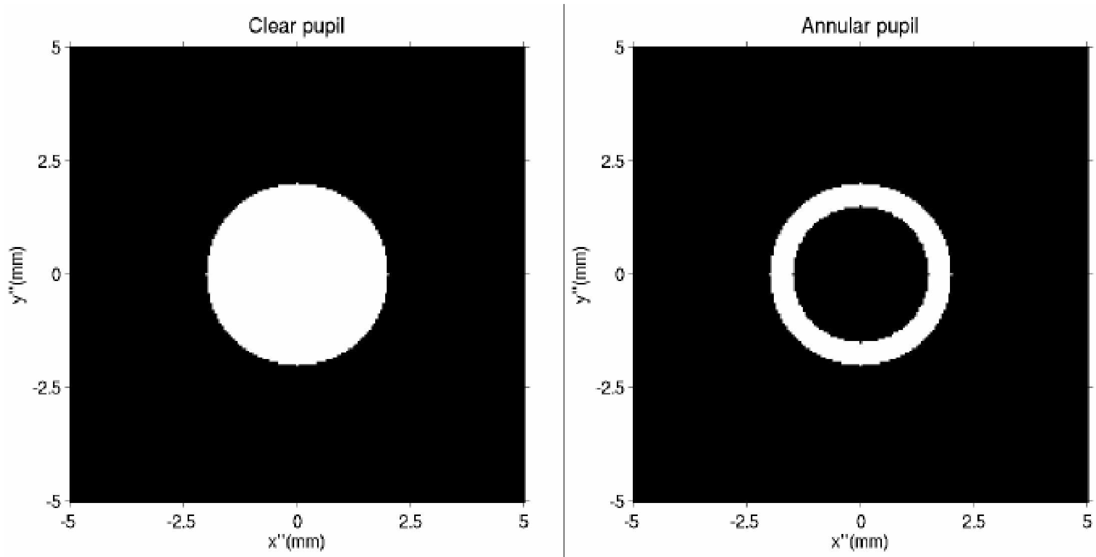
Typical result of optical design



Variation of PSF in the image plane



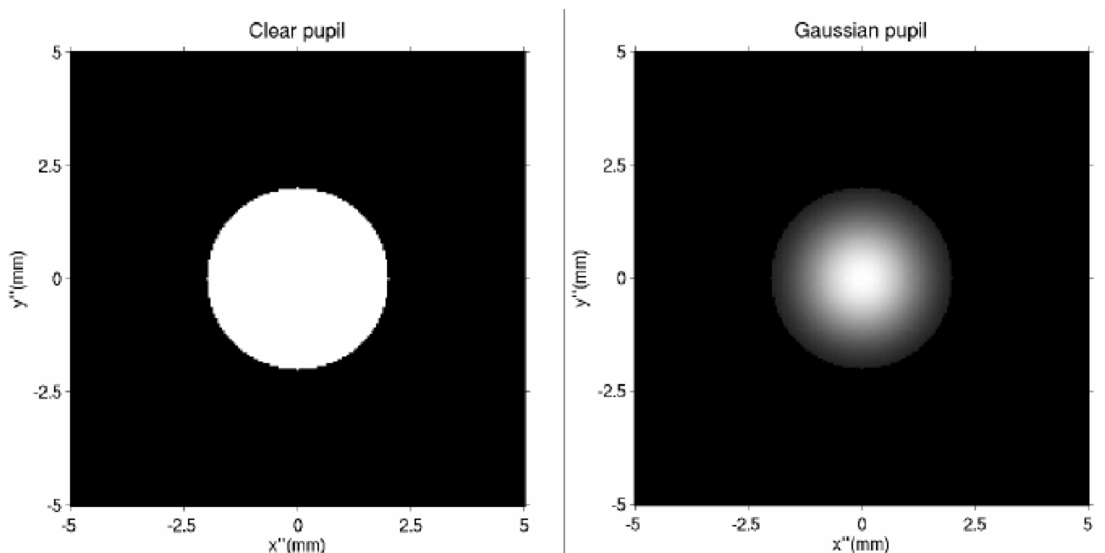
Apodization



f_1=20cm
λ=0.5μm

$$H(r'') = \mathrm{circ}\left(\frac{r''}{R}\right) - \mathrm{circ}\left(\frac{r''}{R_2}\right)$$

Apodization

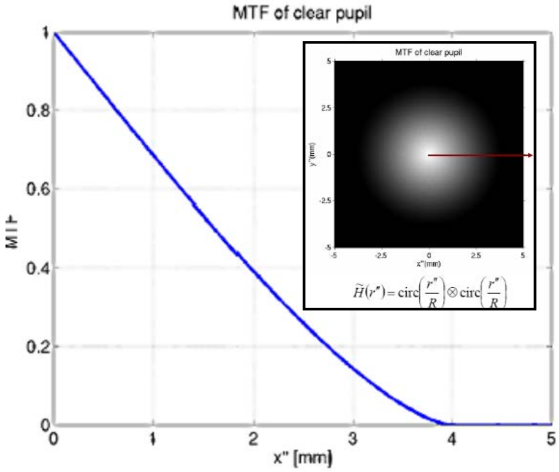


f_1=20cm
λ=0.5μm

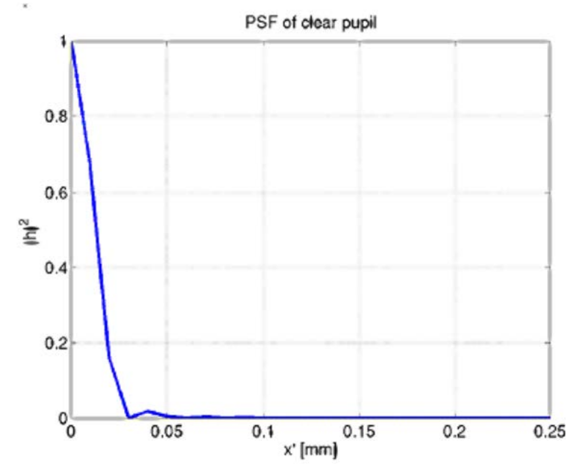
$$H(r'') = \mathrm{circ}\left(\frac{r''}{R}\right) \times \exp\left(-\frac{r''^2}{2R_0^2}\right)$$

For incoherent imaging

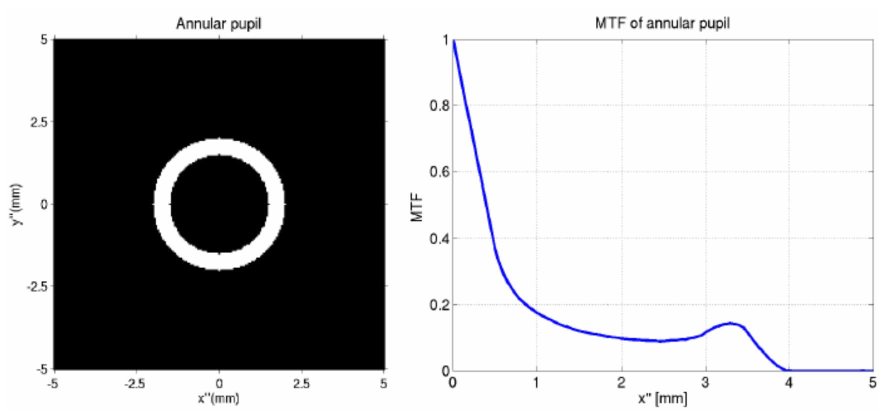
Clear-aperture MTF



iPSF

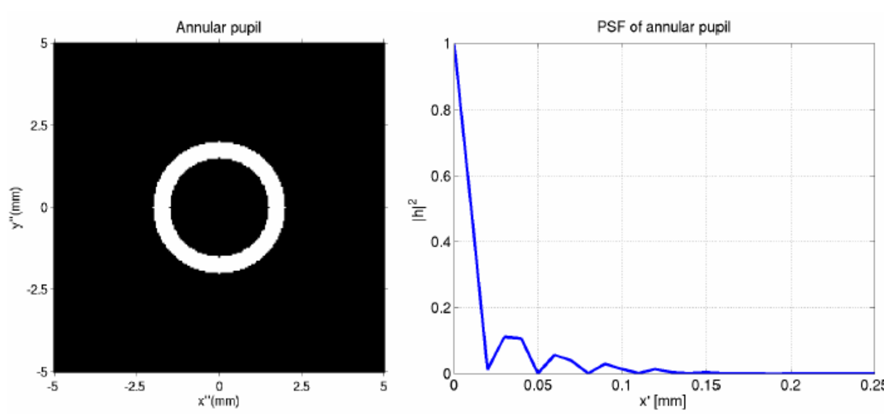


Apodized (annular) MTF



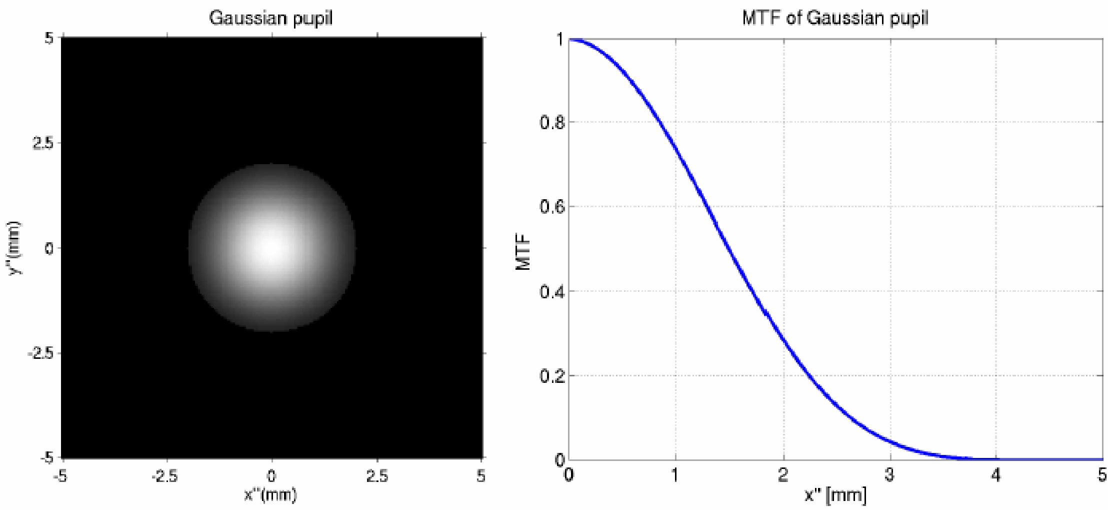
f_1=20cm
λ=0.5μm

Apodized (annular) PSF



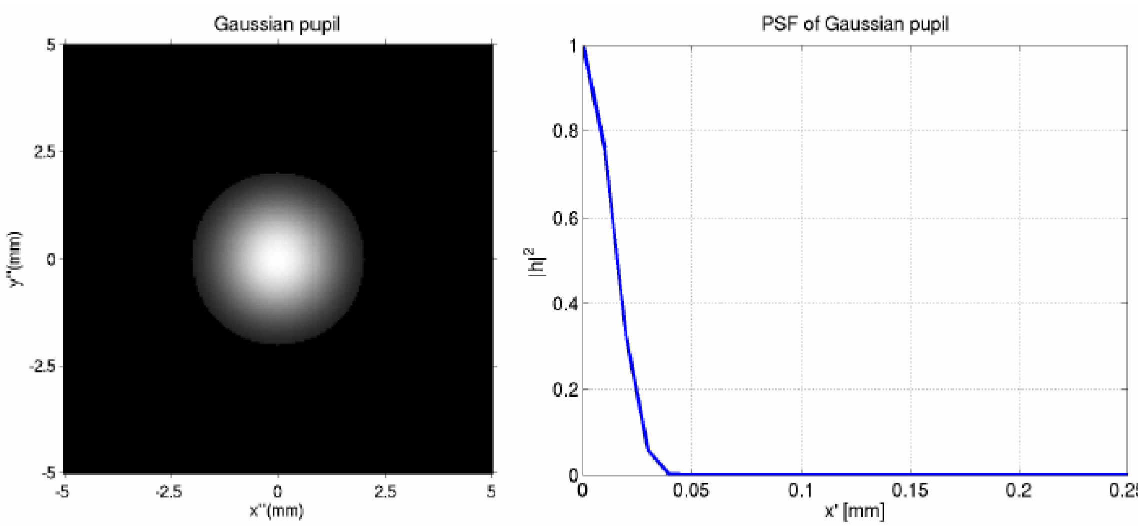
f_1=20cm
λ=0.5μm

Apodized (Gaussian) MTF



f_1=20cm
λ=0.5μm

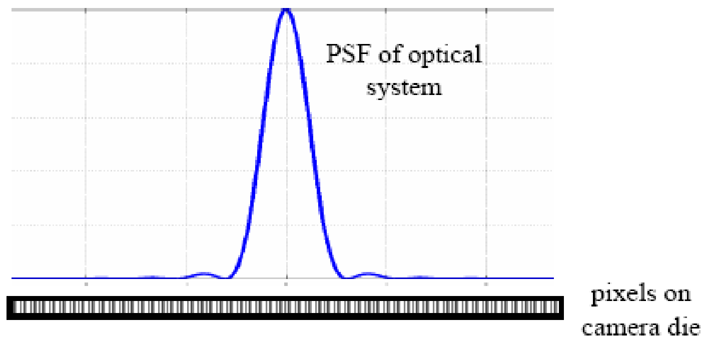
Apodized (Gaussian) PSF



f_1=20cm
λ=0.5μm

What *can* a camera resolve?

Answer depends on the magnification and
PSF of the optical system attached to the camera

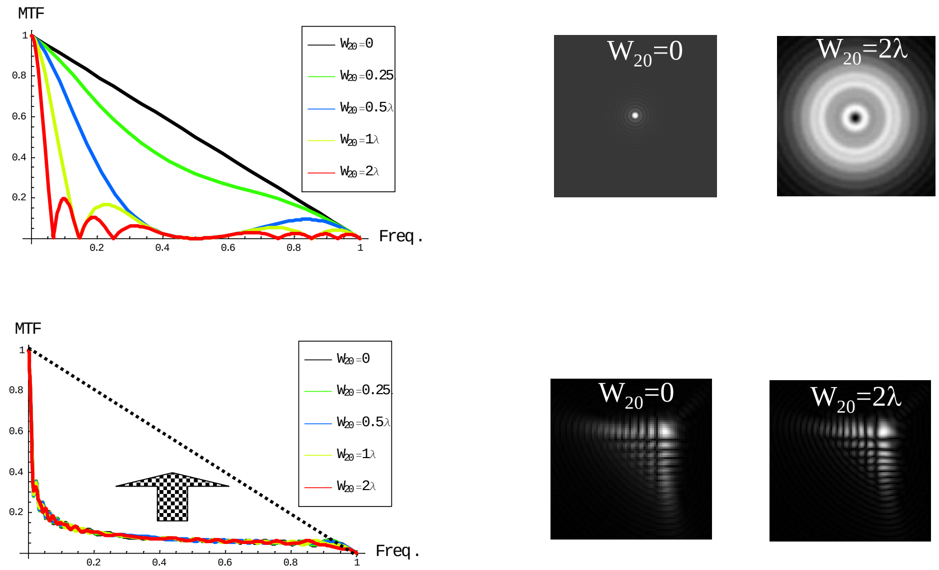


Pixels significantly smaller than the system PSF
are somewhat underutilized (the effective SBP is reduced)

Therefore, it is meaningful to combine image processing function with imaging optics to enhance imaging functionality.

See: Thrufocus.m and ThrufocusEDOF.m

MTF and PSF invariant to defocus via Wavefront Coding



Defocused PSFs observed at different image planes for various phase filters