Chapter 5 Second-Order NLO Effects

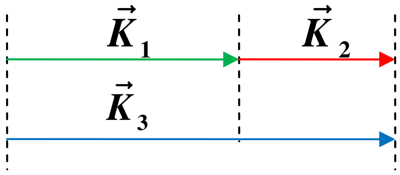
To facilitate the discussion of the topics, we assume all optical media used are weakly nonlinear. ($i.e.$, $\left|\chi^{(2)}E\right| << \left|\chi^{(1)}\right|$), implying that nonlinear effects are observable only when light waves propagate through fairly long distance in a NLO medium. The phase matching condition shall then be fulfilled in order to accumulate the NLO effect:

$$\vec{K}_3 = \vec{K}_1 + \vec{K}_2, \quad i.e.,$$

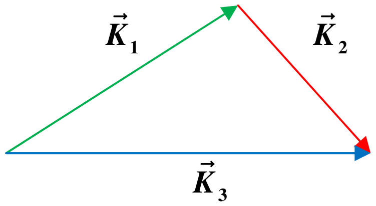
$$\frac{\omega_3 n(\omega_3)}{C}\hat{K}_3 = \frac{\omega_1 n(\omega_1)}{C}\hat{K}_1 + \frac{\omega_2 n(\omega_2)}{C}\hat{K}_2$$

The following two phase matching schemes are usually exploited for this purpose:

- Collinear phase matching scheme



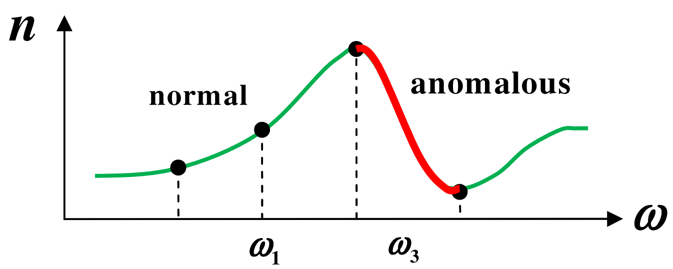
- Noncollinear phase matching scheme



For collinear phase-matched SHG, the corresponding phase matching condition becomes $n_1 = n_3$, which is never fulfilled because of normal dispersion of a material

$n_1(\omega_1 = \omega) < n_3(\omega_3 = 2\omega)$.

The use of anomalous dispersion to meet the phase condition is impractical since the optical energy absorption is fairly high in this regime.



A better way to meet the phase matching condition is to exploit the birefringence of an anisotropic crystal with interaction of differently polarized waves. To illustrate the principle clearly, we will review the essential knowledge of crystal optics in the following section.

5.1 Optics of Uniaxial Crystal

For a uniaxial crystal, there exists

◆ a special direction, called the **optic axis** (the z-axis of the crystallophysical coordinate system);

◆ The plane containing the z-axis and the wave vector $\vec{K}$ of the optical beam is called the **principal plane** (**P.P.**).

If $\vec{E}(\omega) \perp P.P.$, the light beam is called the **ordinary wave** (*o*-wave) of the crystal, which experiences an index of refraction n_o that does not depend on the direction of $\hat{K}$.

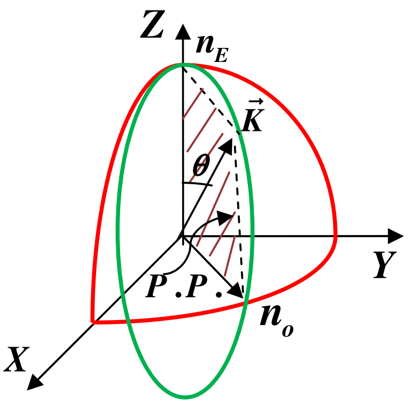
However, when $\vec{E}(\omega) // P.P.$, the light beam is called the **extraordinary wave** (*e*-wave), which experiences $n_e(\theta)$ with magnitude depending on the direction of $\hat{K}$. Let us define

$$\textbf{Birefringence} = \Delta n = n_e(\theta) - n_o = \begin{cases} 0, & when\ \vec{K} // \vec{Z} \\ n_E - n_O = \Delta n, & when\ \vec{K} \perp \vec{Z} \end{cases}$$

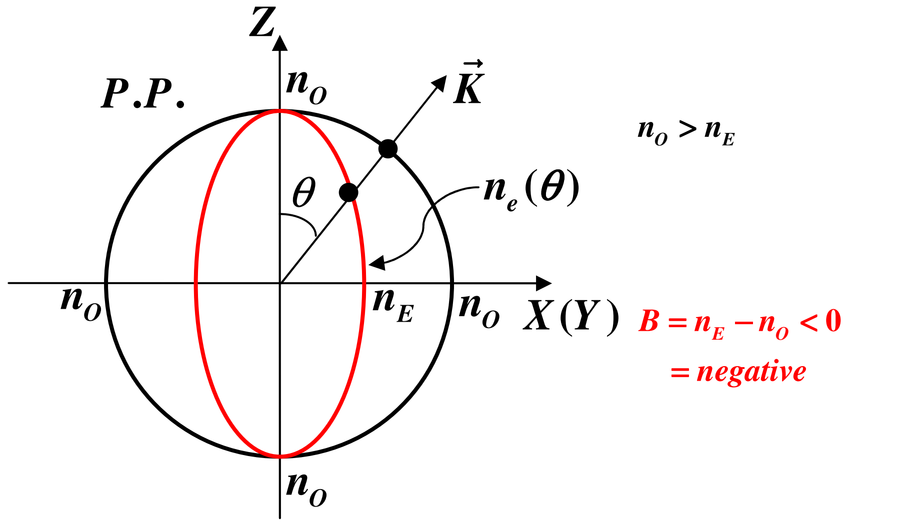
The refractive index of the *e*-wave is a function of the polar angle θ of the vector $\vec{K}$.

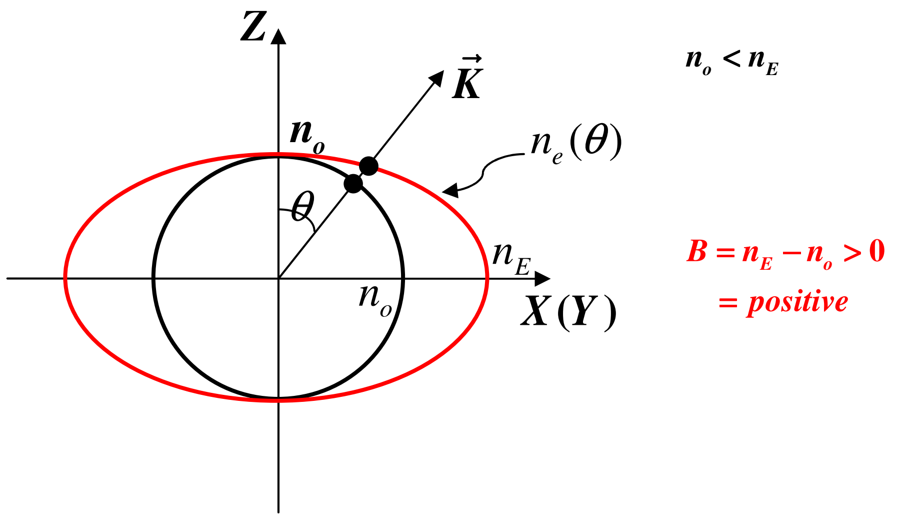
From: $\dfrac{1}{n_e^2(\theta)} = \dfrac{\cos^2\theta}{n_O^2} + \dfrac{\sin^2\theta}{n_E^2}$

$$\Rightarrow \quad n_e(\theta) = n_O \sqrt{\frac{1+\tan^2\theta}{1+(\frac{n_O}{n_E})^2 \tan^2\theta}}$$

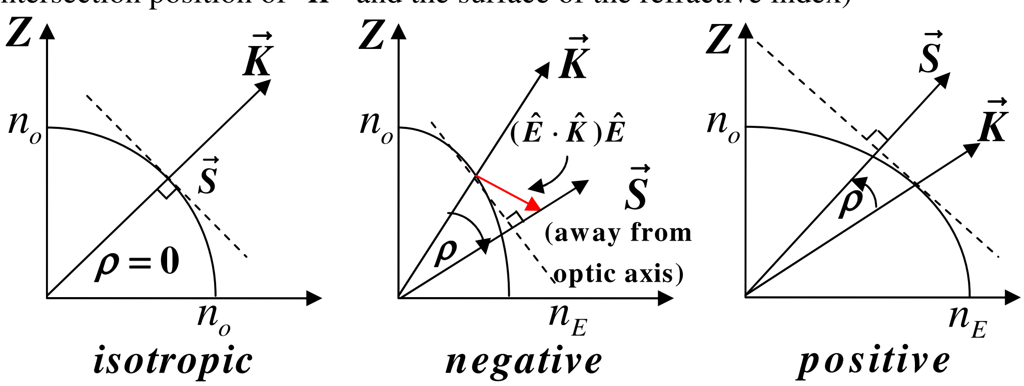


Examining the curves of index of refraction on the principal plane,





For a planar light wave propagating in a uniaxial crystal, $\vec{K}$ does not coincide with that of the wave energy $\vec{S}$ (the unit vector normal to the tangential at the intersection position of $\vec{K}$ and the surface of the refractive index)



Here $\vec{S}$ is normal to the tangential drawn at the point of intersection of vector $\vec{K}$ with the $n_e(\theta)$ curve.

$$\vec{S} = \hat{E} \times \hat{H} = \hat{K} - (\hat{E} \cdot \hat{K})\hat{E}$$

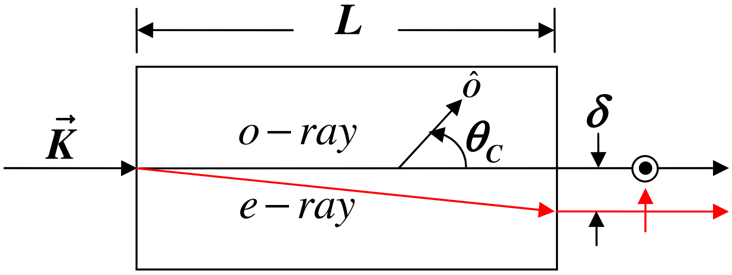
$$\hat{H} = \hat{K} \times \hat{E}$$

E_o $(o-ray)$ always $\perp \vec{K}$ $\Rightarrow$ $\hat{S} // \vec{K}$. But E_e $(e-ray)$ lies in the **P.P.**

$\Rightarrow$ $\hat{S} \not\parallel \vec{K}$, *i.e.*, a beam walk-off phenomenon can be observed. The corresponding walk-off angle ρ can be found to be

$$\rho(\theta) = \pm \tan^{-1}\left[\left(\frac{n_o}{n_E}\right)^2 \tan\theta \right] \mp \theta$$

Upper sign for negative crystals
Lower sign for positive crystals



By measuring the beam walk-off distance δ, we can determine the crystal cutting angle θ_C, which can be understood by noting that

$$\delta = L \cdot \tan\rho \quad \Rightarrow \theta_C = \tan^{-1}\left[\frac{|(n_o/n_E)^2 - 1|L}{2\delta (n_o/n_E)^2} \pm \sqrt{\frac{\left[(n_o/n_E)^2 - 1\right]^2 L^2}{4\delta^2 (n_o/n_E)^4} - (n_o/n_E)^2} \right]$$

6.2 Types of Phase Matching with a Uniaxial Crystal

To fulfill the phase matching condition of a three-wave interaction in a uniaxial crystal, differently polarized waves should be used.

(a) Type-Ⅰ phase matching scheme (Mixing waves have the same polarization)

Negative Crystal: $n_O > n_E$ $(o + o \rightarrow e)$: $\vec{K}_{o1} + \vec{K}_{o2} = \vec{K}_{e3}(\theta)$

Positive Crystal: $n_O < n_E$ $(e + e \rightarrow o)$: $\vec{K}_{e1}(\theta) + \vec{K}_{e2}(\theta) = \vec{K}_{o3}$

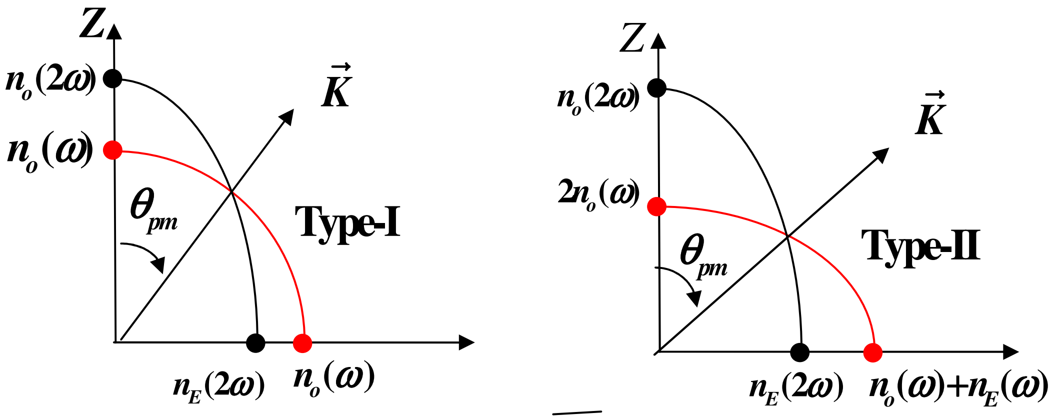
(b) Type-Ⅱ phase matching scheme (the mixing waves have orthogonal polarizations)

Negative Crystal: $\begin{pmatrix} o+e \rightarrow e \\ e+o \rightarrow e \end{pmatrix}$: $\begin{Bmatrix} \vec{K}_{o1} + \vec{K}_{e2}(\theta) = \vec{K}_{e3}(\theta) \\ \vec{K}_{e1}(\theta) + \vec{K}_{o2} = \vec{K}_{e3}(\theta) \end{Bmatrix}$

Positive Crystal: $\begin{pmatrix} o+e \rightarrow o \\ e+o \rightarrow o \end{pmatrix}$: $\begin{Bmatrix} \vec{K}_{o1} + \vec{K}_{e2}(\theta) = \vec{K}_{o3} \\ \vec{K}_{e1}(\theta) + \vec{K}_{o2} = \vec{K}_{o3} \end{Bmatrix}$

For second-harmonic generation, $\omega_1 = \omega_2 = \omega, \quad \omega_3 = 2\omega$

Type-I $o+o \rightarrow e$ interaction in a negative crystal



$$o_1(\omega) + o_2(\omega) \rightarrow e(2\omega)$$
$$n_o(\omega)\frac{\omega}{c} + n_o(\omega)\frac{\omega}{c} = n_e(2\omega,\theta)\frac{2\omega}{c}$$

Type-I: $n_{o1}(\omega) = n_e(2\omega, \theta_{pm})$

Type-II: $n_o(\omega) + n_e(\omega, \theta_{pm}) = 2n_e(2\omega, \theta_{pm})$

6.3 Effective Optical Nonlinearity in a Phase-Matched NLO Process

In the crystallophysical coordinate system {X, Y, Z}, where Z is the optic axis,

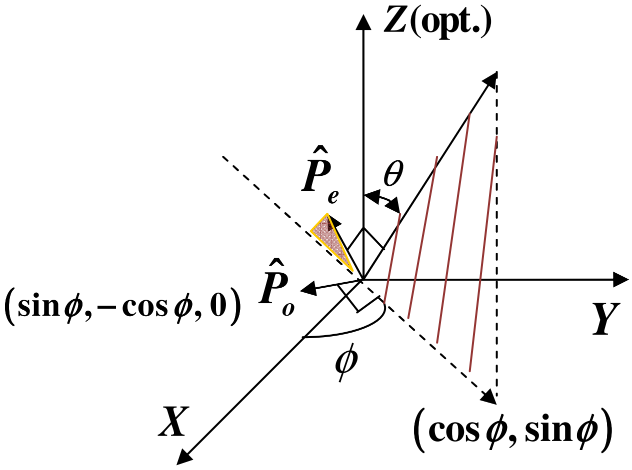
$$P_i^{(2)}(\textbf{In the crystal frame}) = \chi_{ijk}^{(2)} E_j E_k$$

To find out the effective optical nonlinearity observed in the lab frame,

$$P_{eff}^{(2)} = \hat{e}_{out} \cdot \vec{P} = (\hat{e}_{out} \cdot \chi^{(2)} : \hat{e}_{in}\hat{e}_{in})|E_{in}|^2 = \chi_{eff}^{(2)} |E_{in}|^2$$

1×3 3×6 6×1

we can either transform $\chi_{ijk}^{(2)}$ into the lab frame or express $\vec{E}_{in}$ and e_{out} in terms of the crystallophysical coordinate system.



Since any linearly polarized wave in a uniaxial crystal can always be represented as a superposition of two waves with ordinary and extraordinary polarizations, we can express the components of a unit vector $\vec{P}$ given by polar coordinates θ, φ along the crystallophysical axes X, Y, and Z,

$$o\text{-ray} \qquad \vec{P}_o = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}$$

$$e\text{-ray} \qquad \vec{P}_e = \begin{pmatrix} -\cos\theta\cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta \end{pmatrix}$$

therefore, $\chi_{eff}^{(2)} = \sum_{ijk} (\hat{P}_{out})_i \cdot \left[\chi_{ijk}^{(2)} : (\vec{P}_{in})_j (\vec{P}_{in})_k \right]$.

We illustrate the principle with some case studies:

Example 1

For KDP ($\bar{4}2m$) in Type- I PM $(o+o\rightarrow e)$

$$\chi_{XYZ}^{(2)} = \chi_{YXZ}^{(2)} \cong \chi_{XZY}^{(2)} = \chi_{YZX}^{(2)} \cong 2.6\times10^{-9}\,esu$$
$$\chi_{ZXY}^{(2)} = \chi_{ZYX}^{(2)} = 2.82\times10^{-9}\,esu$$

$$(P_{o1}P_{o2})_{contracted} = \begin{pmatrix} \sin^2\phi \\ \cos^2\phi \\ 0 \\ 0 \\ 0 \\ -\sin\phi\cos\phi \end{pmatrix}$$

For Type- I PM $\overset{1}{(o} + \overset{2}{o} \to \overset{3}{e})$

$$\chi_{eff}^{(2)} = \hat{e}_3 \cdot \overleftrightarrow{\chi} : (\hat{O}_1\hat{O}_2)_{contracted}$$
$$= -\chi_{ZXY}^{(2)} \sin\theta \sin 2\phi$$
$$\cong -2.74\times10^{-9}\, esu$$

Note: allow $\sin 2\phi$ to be max. with $\phi = 45^0$.

$$\overleftrightarrow{\chi}(\bar{4}2m) = \begin{pmatrix} 0 & 0 & 0 & r_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{36} \end{pmatrix}$$

Example 2

(***$C_{\infty V}$ poled polymer***)

$$\overleftrightarrow{\chi}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \chi_{15} & 0 \\ 0 & 0 & 0 & \chi_{15} & 0 & 0 \\ \chi_{31} & \chi_{31} & \chi_{33} & 0 & 0 & 0 \end{pmatrix}, \quad \chi_{31} = \chi_{15} \quad \textit{Kleinman Symmetry}$$

(i) For Type-I PM (negative)

$$\chi_{eff} = \hat{e}_3 \cdot \overleftrightarrow{\chi}^{(2)} : (\hat{o}_1\hat{o}_2)_{contracted}$$

$$= (-\cos\phi\cos\theta, -\sin\phi\cos\theta, \sin\theta)[\chi^{(2)}] \begin{pmatrix} \sin^2\phi \\ \cos^2\phi \\ 0 \\ 0 \\ 0 \\ -\sin\phi\cos\phi \end{pmatrix} = \chi_{15}\sin\theta$$

(ii) For Type-I PM (positive)

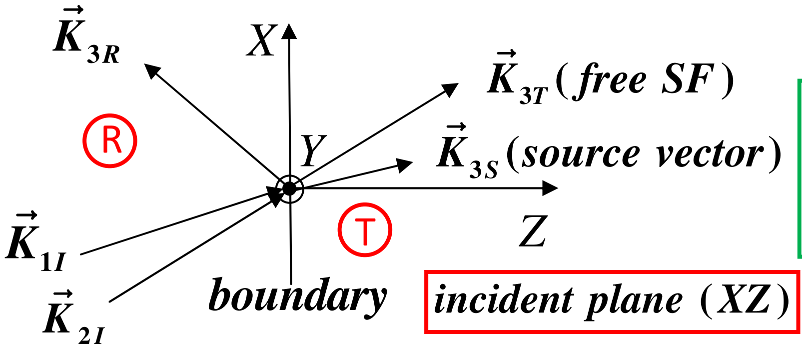
$$\chi_{eff} = \hat{o}_3 \cdot \overleftrightarrow{\chi}^{(2)} : (\hat{e}_2\hat{e}_1)_{contracted} = 0$$

χ_{33} does not contribute to *PM SHG*

6.4 Sum-Frequency Generation with Boundary Reflection

[**Ref.:** N. Bloembergen and P. S. Pershan, Phys. Rev. 128, 606-622 (1962)]

Consider $\omega_1 + \omega_2 \rightarrow \omega_S = \omega_3 = \omega_1 + \omega_2$



$$P_{NL}^{(2)}(\omega_3) = \chi^{(2)} E_{1T} E_{2T} \cdot e^{i(\vec{K}_{1T}+\vec{K}_{2T})\cdot r - i(\omega_1+\omega_2)t} = \chi^{(2)} E_{1T} E_{2T} \cdot e^{i\vec{K}_{3S}\cdot r - i\omega_3 t}$$

To make the problem solvable, we are going to make some assumptions

- All waves involved are plane waves
- Non-depleted incident waves
- Planar boundary (**XY** plane)
- Neglect anisotropic propagation property of the nonlinear medium

$$\nabla\times[\nabla\times E(\omega_3)] - \frac{E(\omega_3)\omega_3^2}{c^2}\cdot E(\omega_3) = \frac{4\pi\omega_3^2}{c^2} P^{NLS}\cdot e^{i\vec{K}_S\cdot\vec{r}-i\omega_3 t} = 4\pi\tilde{\omega}_3^2 P^{NLS}\cdot e^{i\vec{K}_S\cdot\vec{r}-i\omega_3 t} \quad .$$

Here $\tilde{\omega}_3 = \frac{\omega_3}{c}$, $\vec{K}_{3S} = \vec{K}_S$, $P^{NLS} = \chi^{(2)} E_{1T} E_{2T}$. The complete solution of the above equation can be expressed as a summation of homogeneous solution and particular solution with

- Homogeneous solution = Linear combination of a set of plane waves of free SF with the coefficients to be determined by the boundary conditions
- Particular solution denotes the bound SF waves with $\vec{K}_{SX}$, $\vec{K}_{SY}$ determined by the direction of the incident beams $\vec{K}_1^I$, $\vec{K}_2^I$. The constraint is due to that on reflection and refraction, the transverse components (tangential to the interface) of momentum are conserved because the boundary conditions must be satisfied everywhere on the plane **Z = 0**.

$$K_{SX} = K_{1X}^I + K_{2X}^I = K_{1X}^T + K_{2X}^T = K_{1X} + K_{2X}$$
$$K_{SY} = K_{1Y}^I + K_{2Y}^I = K_{1Y}^T + K_{2Y}^T = K_{1Y} + K_{2Y}$$

After entering into the nonlinear medium ($Z \geq 0$), the incident waves induce a nonlinear polarization P^{NLS} . The radiation source can emit SF photons at ω_3 .

However, only those radiated free waves (assume to be planar waves) which have the same tangential wave vector components are acceptable to be included to match the boundary conditions. This determines the direction of the free waves, acceptable as homogeneous solution.

The direction of the inhomogeneous solution or polarization wave is determined by its normal component because $\vec{K}_S = \vec{K}_1^T + \vec{K}_2^T$ and from

$$K_{SX} = K_{1X}^T + K_{2X}^T \;\; and \;\; K_{SY} = K_{1Y}^T + K_{2Y}^T$$
$$\Rightarrow \quad K_{SZ} = \sum_i K_{iZ}^T = K_{1Z}^T + K_{2Z}^T$$

T: normal components transmitted into the NLO medium

Let the incident plane is XZ plane, i.e. $K_{SY} = 0$.

Let us distinguish the following two linearly independent cases with (A) $P_Y^{NLS} = P_\perp^{NLS}$ and (B) $P_{//}^{NLS}$ (on the XZ plane).

(A) $\mathrm{P}_\perp^{NLS} = P_\perp^{NLS} e^{i\vec{K}_S \cdot \vec{r}} = (\chi_{eff}^{(2)})_{yij} E_{1i}^T E_{2j}^T e^{i(\vec{K}_1^T + \vec{K}_2^T)\cdot \vec{r}}$. The wave eq. becomes

$$\nabla^2 E_\perp + \tilde{\omega}_3^2 \varepsilon_T(\omega_3) E_\perp = \nabla^2 E_\perp + |K_T|^2 E_\perp = -4\pi\tilde{\omega}_3^2 P_\perp^{NLS} e^{i\vec{K}_S \cdot \vec{r}}.$$

Let the particular solution be $E_\perp = Ae^{i\vec{K}_S \cdot \vec{r}}$ and substitute into the above equation

$$[-|K_S|^2 + |K_T|^2] \cdot A \cdot e^{i\vec{K}_S \cdot \vec{r}} = -4\pi\tilde{\omega}_S^2 P_\perp^{NLS} e^{i\vec{K}_S \cdot \vec{r}}$$
$$\Rightarrow A = \frac{4\pi\tilde{\omega}_3^2}{|K_S|^2 - |K_T|^2} \cdot P_\perp^{NLS}.$$

The complete solution then becomes

$$E_\perp = A_\perp^T e^{i\vec{K}^T \cdot \vec{r}} + \frac{4\pi\tilde{\omega}_3^2 P_\perp^{NLS}}{|K_S|^2 - |K_T|^2} \cdot e^{i\vec{K}_S \cdot \vec{r}}$$
$$E_\perp \; perpendicular \; to \; \vec{K}^T$$
$$|\vec{K}^T| = \omega_3 \frac{\sqrt{\varepsilon_T(\omega_3)}}{c} = \tilde{\omega}_3 \sqrt{\varepsilon_T} = n_T \tilde{\omega}_3$$

$\vec{K}^T$ represents the transmitted homogeneous solution with $K_X^T = K_{SX}$.

(ii) $A_\perp^T$ is determined by the requirement that tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous.

(iii) However, to meet these boundary conditions, a reflected wave at ω_3, emanating from the boundary back into the linear medium shall be included,

$$E_\perp^R = A_\perp^R e^{i\vec{K}^R\cdot\vec{r}} \quad with \;\; K_X^R = K_X^T = K_{SX} \;\; \text{and} \;\; \left|K^R\right| = \frac{\omega_3}{c} n_R(\omega_3) = \tilde{\omega}_3 n_R$$

Thus $\boxed{A_\perp^T + \dfrac{4\pi\tilde{\omega}_3^2 P_\perp^{NLS}}{|K_S|^2 - |K_T|^2} = A_\perp^R}$ from $\boxed{E_Y\,|_{Z=0} = E_Y^R\,|_{Z=0}}$

$$\boxed{\tilde{\omega}_3 n_T A_\perp^T \cos\theta_T + \frac{|K_S|\,4\pi\tilde{\omega}_3^2 P_\perp^{NLS}}{|K_S|^2 - |K_T|^2}\cos\theta_S = -\tilde{\omega}_3 n_R A_\perp^R \cos\theta_R}.$$

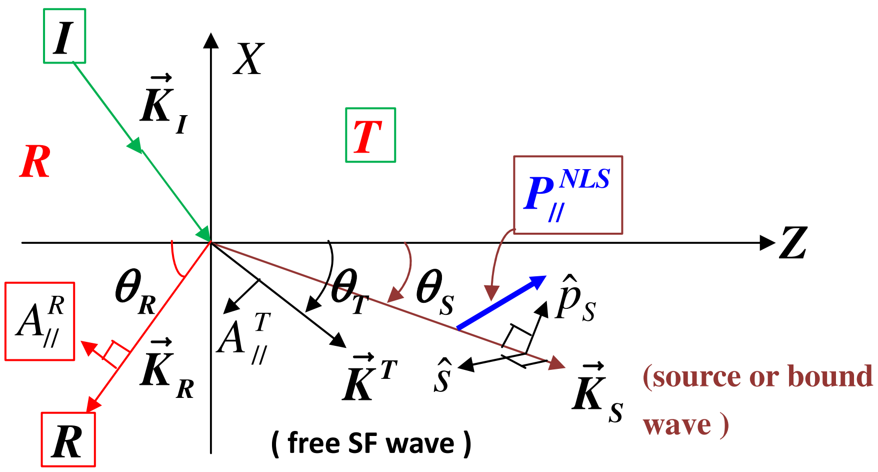
Note that $\boxed{\cos\theta_S \;\; \text{refers to} \;\; \vec{K}_S}$ and $\boxed{\cos\theta_R \;\; \text{refers to} \;\; \vec{K}^R}$.

Furthermore, $\boxed{from\ H_X - continuity}$ $A_\perp^T$ and $A_\perp^R$ can be determined

$$A_\perp^R = -4\pi P_\perp^{NLS}\cdot\frac{1}{(n_T\cos\theta_T + n_R\cos\theta_R)(n_T\cos\theta_T + n_S\cos\theta_S)} \qquad where \;\; n_S = \frac{c|K_S|}{\omega_3}$$

$$A_\perp^T = \frac{4\pi P_\perp^{NLS}}{\varepsilon_T - \varepsilon_S}\cdot\frac{n_S\cos\theta_S + n_R\cos\theta_R}{n_T\cos\theta_T + n_R\cos\theta_R}$$

(B) Parallel Polarization $\vec{P}_{//}^{NLS}$ (on XZ plane)



Note:

$$\nabla\cdot\vec{D}=\nabla\cdot[\underbrace{\vec{E}_{//}+4\pi\vec{P}_{//}^{(1)}}_{=\varepsilon_T E_{//}}\cdot\hat{K}+4\pi\vec{P}_{//}^{(2)}\cdot\hat{K}]=0$$

parallel to $\vec{K}$

$$\Rightarrow \vec{E}_{//}(\omega_3)=\frac{4\pi\vec{P}_{//}^{(2)}}{\varepsilon_T(\omega_3)}$$

E_X *– component* :

$$A_{//}^{T}\cos\theta_T+\frac{4\pi\tilde{\omega}_3^2(\vec{P}_{//}^{NLS}\cdot\hat{p}_S)\cos\theta_S}{|K_S|^2-|K_T|^2}-\frac{4\pi(\vec{P}_{//}^{NLS}\cdot\hat{K}_S)\sin\theta_S}{\varepsilon_T}=-A_{//}^{R}\cos\theta_R$$

H_Y *– component* :

$$K^TA_{//}^T+K_S\frac{4\pi\tilde{\omega}_3^2(\vec{P}_{//}^{NLS}\cdot\hat{p}_S)}{|K_S|^2-|K_T|^2}=K^RA_{//}^R=(K^R\times A^R)_Y$$

By solving the simultaneous equations, we obtain

$$A_{//}^{T}=\frac{4\pi}{n_R\cos\theta_T+n_T\cos\theta_R}\cdot\left\{\frac{(\vec{P}_{//}^{NLS}\cdot\hat{K}_S)\sin\theta_R n_R}{\varepsilon_T}-\frac{(n_R\cos\theta_R+n_S\cos\theta_S)(\vec{P}_{//}^{NLS}\cdot\hat{p}_S)}{(\varepsilon_S-\varepsilon_T)}\right\}$$

Parallel to $\vec{K}$ and on the incident plane

Perpendicular to $\vec{K}$

$$A_{//}^{R}=\frac{4\pi}{n_R\cos\theta_T+n_T\cos\theta_R}\cdot\left\{\frac{(\vec{P}_{//}^{NLS}\cdot\hat{K}_S)\sin\theta_S}{\varepsilon_T}-\frac{(n_T\cos\theta_S-n_S\cos\theta_T)(\vec{P}_{//}^{NLS}\cdot\hat{p}_S)}{(\varepsilon_S-\varepsilon_T)}\right\}$$

6.5 Bulk Sum-Frequency Generation

For the case of sum-frequency generation from the bulk of a nonlinear medium, we can exploit the slowly varying amplitude approximation (SVA). This leads to

1. P^{NLS} is perpendicular to the incident plane

$$\frac{\partial}{\partial z}E_\perp(Z)=\frac{i2\pi\tilde{\omega}_3^2}{K_Z^T}P_\perp^{NLS}e^{i\Delta K Z},\qquad \text{where } \Delta K=K_S^T-K_3^T=(K_1^T+K_2^T)-K_3^T$$

2. P^{NLS} lies in the incident plane

$$\frac{\partial}{\partial z}[\varepsilon_T E_{//} + 4\pi(P_{//}^{NLS} \cdot \hat{K}_S)e^{i\Delta K Z}] = 0$$

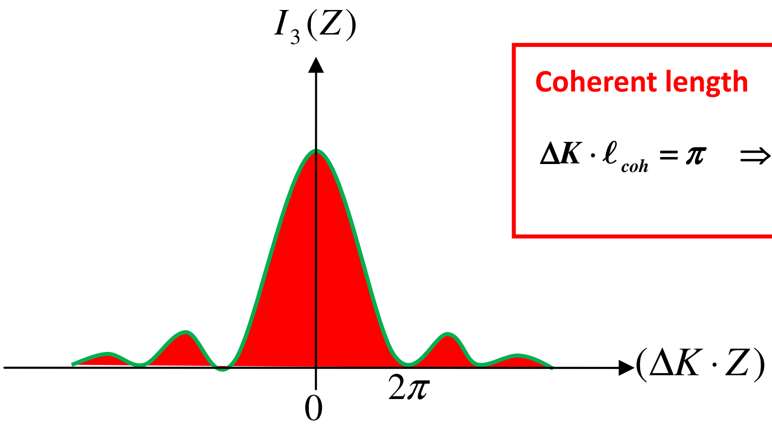
$$\therefore \begin{cases} E_\perp^T(z) = E_\perp(0) + \dfrac{2\pi\tilde{\omega}_3^2}{(\Delta K)K_Z^T}(e^{i\Delta K z} - 1)P_\perp^{NLS} \\ E_{//}(z) = E_{//}(0) - \dfrac{4\pi(\vec{P}_{//} \cdot \hat{K}_S)}{\varepsilon_T}(e^{i\Delta K z} - 1) \end{cases}$$

Note:

If $\left|\dfrac{K_3}{\Delta K}\right| >> 1$, then $|E_{//}| << |E_\perp|$

$$I_3(z) = \frac{c\sqrt{\varepsilon_T}}{2\pi}|E(z)|^2 \quad (\text{Note: } K_Z^T = \frac{\omega_3\sqrt{\varepsilon_T}}{c}\cos\theta_T)$$

$$= \frac{2\pi\omega_3^2}{c\sqrt{\varepsilon_T}\cos\theta_T} \cdot |P_\perp|^2 \cdot \left[\frac{\sin(\Delta K \cdot \frac{z}{2})}{\Delta K \cdot \frac{z}{2}}\right]^2 \cdot z^2$$



Coherent length

$$\Delta K \cdot \ell_{coh} = \pi \quad \Rightarrow \quad \ell_{coh} = \frac{\pi}{\Delta K}$$

6.6 Sum-Frequency Generation with High Conversion Efficiency

For sum-frequency generation with high conversion efficiency, the following conditions must be fulfilled:

(1) the coupled waves are collinearly phase matched ($\Delta K = 0$)

(2) the medium is lossless ($\sigma_n = 0$)

(3) slowly varying amplitude (SVA)

The incident waves will be depleted due to strong conversion that requires the three coupled differential equations to be solved simultaneously.

$$\begin{cases} \dfrac{\partial E_1}{\partial Z} \overset{m.k.s}{=} \dfrac{i\omega_1^2}{c^2 K_{1Z}\cos^2\alpha_1} d_1 E_2^* E_3 \overset{esu}{=} \dfrac{2\pi i\tilde{\omega}_1^2}{\cos^2\alpha_1 K_{1Z}} \cdot \chi_{eff}^{(2)} : E_2^* E_3 \\ \dfrac{\partial E_2}{\partial Z} \overset{m.k.s}{=} \dfrac{i\omega_2^2}{c^2 K_{2Z}\cos^2\alpha_2} d_2 E_1^* E_3 \overset{esu}{=} \dfrac{2\pi i\tilde{\omega}_2^2}{\cos^2\alpha_2 K_{2Z}} \cdot \chi_{eff}^{(2)} : E_1^* E_3 \\ \dfrac{\partial E_3}{\partial Z} \overset{m.k.s}{=} \dfrac{i\omega_3^2}{c^2 K_{3Z}\cos^2\alpha_3} d_3 E_1 E_2 \overset{esu}{=} \dfrac{2\pi i\tilde{\omega}_3^2}{\cos^2\alpha_3 K_{3Z}} \cdot \chi_{eff}^{(2)} : E_1 E_2 \end{cases}$$

where α_i = walk off angle of beam i, and

$$\chi_{eff,1}^{(2)} = \hat{e}_1 \cdot \chi^{(2)} : \hat{e}_2\hat{e}_3$$
$$\chi_{eff,2}^{(2)} = \hat{e}_2 \cdot \chi^{(2)} : \hat{e}_3\hat{e}_1$$
$$\chi_{eff,3}^{(2)} = \hat{e}_3 \cdot \chi^{(2)} : \hat{e}_1\hat{e}_2$$

In a lossless medium, $d_1 = d_2 = d_3 = d$ (why?).

Let $\begin{matrix} \alpha_1 = \alpha_2 = 0 & (\ Type-\text{I}\ \ negative\ \ crystal\ ) \\ \alpha_3 = \beta = walk\ off\ angle & (\underset{1}{o} + \underset{2}{o} \to \underset{3}{e}) \end{matrix}$ and define

$$W = \frac{c^2}{2\pi}\left\{ \frac{K_{1Z}|E_1|^2}{\omega_1} + \frac{K_{2Z}|E_2|^2}{\omega_2} + \frac{K_{3Z}\cos^2\beta|E_3|^2}{\omega_3} \right\}$$
$$= \frac{c}{2\pi}\left\{ n_1|E_1|^2 + n_2|E_2|^2 + n_3\cos^2\beta|E_3|^2 \right\} = \text{constant, which is } independent\ of\ Z$$

This is the **Manley-Rowe Relation**, implying that the number of photons annihilated **at** ω_1 **and** ω_2 = the number of photons created at ω_3

For convenience, we can use $\sqrt{W}$ to normalize the optical field amplitudes

$$\sqrt{\frac{I_i}{\omega_i W}} e^{i\phi_i} = \sqrt{\frac{c^2 K_{1Z}}{2\pi\omega_1^2 W}} E_1(Z) = u_i e^{i\phi_i}$$ **= square root of the normalized photon density at frequency** ω_i

Thus $u_1 e^{i\phi_1} = \sqrt{\dfrac{I_1}{\omega_1 W}} e^{i\phi_1}, \quad u_2 e^{i\phi_2} = \sqrt{\dfrac{I_2}{\omega_2 W}} e^{i\phi_2}, \quad u_3 e^{i\phi_3} = \sqrt{\dfrac{I_3}{\omega_3 W}} e^{i\phi_3}$

with $\theta(Z) = \phi_3(Z) - \phi_1(Z) - \phi_2(Z)$

Properly normalize the propagation distance Z with Z_I to yield a dimensionless distance-related parameter

$$\boxed{\xi = \sqrt{\frac{2\pi W\,\omega_1^2\omega_2^2\omega_3^2}{c^2 K_{1Z}K_{2Z}K_{3Z}\cos^2\beta}}\cdot\frac{2\pi}{c^2}\chi_{eff}^{(2)}\cdot Z} = \frac{Z}{Z_I}$$

Finally

$$\begin{cases} \dfrac{du_1}{d\xi} = -u_2u_3\sin\theta & (1)\\ \dfrac{du_2}{d\xi} = -u_3u_1\sin\theta & (2)\\ \dfrac{du_3}{d\xi} = u_1u_2\sin\theta & (3)\\ \boxed{\dfrac{d\theta}{d\xi} = \left(\dfrac{u_1u_2}{u_3} - \dfrac{u_2u_3}{u_1} - \dfrac{u_1u_3}{u_2}\right)\cos\theta} & (4) \end{cases}$$

By using Eqs. (1), (2) and (3), Eq. (4) can be reduced to

$$\begin{aligned} \frac{d\theta}{d\xi} &= \left(\frac{1}{\sin\theta}\frac{1}{u_3}\frac{du_3}{d\xi} + \frac{1}{\sin\theta}\frac{1}{u_1}\frac{du_1}{d\xi} + \frac{1}{\sin\theta}\frac{1}{u_2}\frac{du_2}{d\xi}\right)\cos\theta \\ &= \cot\theta\cdot\frac{d}{d\xi}\ell n(u_1u_2u_3) \end{aligned}$$

This is a complete differential, we therefore can simply integrate the phase equation to yield $\tan\theta\,d\theta = d\ \ell n(u_1u_2u_3) \Rightarrow\ -d\left[\ell n\cos\theta\right] = d\,\ell n(u_1u_2u_3)$.

This leads to $\boxed{\cos\theta(Z) = \dfrac{\Gamma}{(u_1u_2u_3)}}$ with an integration constant Γ.

Several invariants can also be found, including

(i) $\Gamma = u_1(0)u_2(0)u_3(0)\cos\theta(0)$ with $\theta(0) = \phi_3(0) - \phi_1(0) - \phi_2(0)$;

(ii) Energy conservation: $\omega_3 = \omega_1 + \omega_2$,

(iii) power normalization $\omega_1u_1^2 + \omega_2u_2^2 + \omega_3u_3^2 = 1$ and

$$
\begin{aligned}
&u_1 \times \text{Eq.}(1) + u_3 \times \text{Eq.}(3) \quad \Rightarrow \quad m_1 = u_1^2 + u_3^2 \\
\text{(iv)}\quad &u_2 \times \text{Eq.}(2) + u_3 \times \text{Eq.}(3) \quad \Rightarrow \quad m_2 = u_2^2 + u_3^2 \\
&u_1 \times \text{Eq.}(1) - u_2 \times \text{Eq.}(2) \quad \Rightarrow \quad m_3 = u_1^2 - u_2^2
\end{aligned}
$$

The invariants reveal the Manley-Rowe Relations

From $\dfrac{du_3}{d\xi} = u_1 u_2 \sin\theta$, we rewrite

$$
d\xi = \frac{du_3}{u_1 u_2 \sin\theta} = \frac{du_3}{u_1 u_2 \sqrt{1 - \dfrac{\Gamma^2}{(u_1 u_2 u_3)^2}}}
$$

$$
\Rightarrow \quad \xi = \frac{1}{2} \int_{u_3^2(0)}^{u_3^2(\xi)} \frac{d(u_3^2)}{\sqrt{u_3^2 (m_2 - u_3^2)(m_1 - u_3^2) - \Gamma^2}},
$$

The denominator polynomial can be rewritten as $(u^2 - u_{3c}^2)(u^2 - u_{3b}^2)(u^2 - u_{3a}^2)$ in terms of its roots $u_{3c}^2 \geq u_{3b}^2 \geq u_{3a}^2 \geq 0$.

By defining $y^2 = \dfrac{u_3^2 - u_{3a}^2}{u_{3b}^2 - u_{3a}^2}$, $\gamma = \dfrac{u_{3b}^2 - u_{3a}^2}{u_{3c}^2 - u_{3a}^2}$, we then obtain

$$
\xi = \frac{1}{\sqrt{u_{3c}^2 - u_{3a}^2}} \cdot \left\{ \int_0^{y(\xi)} \frac{dy}{\sqrt{(1-y^2)(1-\gamma^2 y^2)}} - \int_0^{y(0)} \frac{dy}{\sqrt{(1-y^2)(1-\gamma^2 y^2)}} \right\}.
$$

The integrals can be related to the Jacobian elliptic functions, which are defined as follows

$$
u(\sin\phi, \gamma) \equiv F(\phi, \gamma) \equiv \int_0^{\sin\phi} \frac{dy}{\sqrt{(1-y^2)(1-\gamma^2 y^2)}}.
$$

By taking the inverse function $Sn(u,\gamma) = \sin\phi$ of $u(\sin\phi, \gamma)$,

i.e., $u = \displaystyle\int_0^{Sn} \frac{dy}{\sqrt{(1-y^2)(1-\gamma^2 y^2)}}$

Note: $\gamma^2 (Sn)^2 + (dn)^2 = 1$. **When** $\gamma = 0$, $Sn(u, \gamma = 0) = \sin u$.

In terms of the Jacobian elliptic function Sn, the solution becomes

$$
y(\xi) = Sn\left[\sqrt{(u_{3c}^2 - u_{3a}^2)} (\xi + \xi_0), \gamma \right].
$$

Note that $u_3^2 = u_{3a}^2 + (u_{3b}^2 - u_{3a}^2) y^2$, we then obtain the solution for the three-wave mixing processes with depleted pump beam as

$$\begin{cases} u_3^2(\xi) = u_{3a}^2 + (u_{3b}^2 - u_{3a}^2) Sn^2\left[\sqrt{(u_{3c}^2 - u_{3a}^2)}(\xi + \xi_0), \gamma\right] \\ u_2^2(\xi) = m_2 - u_3^2(\xi) = u_2^2(0) + u_3^2(0) - u_3^2(\xi) \\ u_1^2(\xi) = u_1^2(0) + u_3^2(0) - u_3^2(\xi) \end{cases}$$

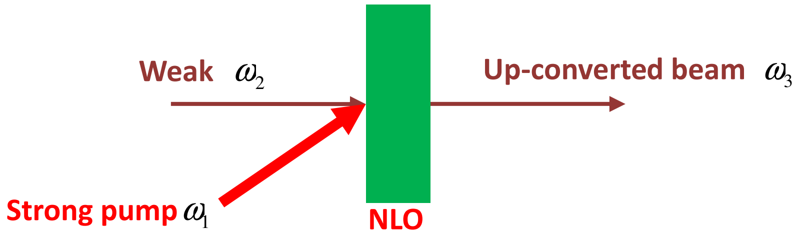
.

where $\xi_0 = \dfrac{1}{\sqrt{u_{3c}^2 - u_{3a}^2}} F(\sin^{-1} y(0), \gamma)$

.

Some examples

1. Frequency Up-conversion



Input conditions:

$$\begin{aligned} &frequencies: \quad \omega_1 + \omega_2 \rightarrow \omega_3 \\ &input\ fields \quad u_1(0) + u_2(0) \rightarrow u_3(0) \\ &\qquad\qquad u_2^2(0) << u_1^2(0) \\ &\qquad\qquad u_3^2(0) \cong 0 \\ &\Rightarrow \quad \Gamma = u_1(0)u_2(0)u_3(0)\cos\theta(0) = 0 \end{aligned}$$

The three roots of $u_3^2(m_2 - u_3^2)(m_1 - u_3^2) - \Gamma^2 = 0$ **are** $u_3 = \sqrt{m_1},\ \sqrt{m_2},\ 0.$

Therefore, $\boxed{u_{3a} = 0,\ u_{3b} = \sqrt{m_2} = u_2(0),\ u_{3c} = \sqrt{m_1} = u_1(0)\,.}$

The solutions become:

$$u_3^2(\xi) = u_2^2(0)Sn^2\left[u_1(0)(\xi + \xi_0), \gamma\right]$$

$$u_2^2(\xi) = u_2^2(0)\left[1 - Sn^2\left[u_1(0)(\xi + \xi_0), \gamma\right]\right]$$

$$u_1^2(\xi) = u_1^2(0) - u_2^2(0)Sn^2\left[u_1(0)(\xi + \xi_0), \gamma\right]$$

$$Since \quad \gamma^2 = \frac{(u_{3b}^2 - u_{3a}^2)}{(u_{3c}^2 - u_{3a}^2)} = \frac{u_2^2(0)}{u_1^2(0)} << 1$$

$$Therefore \quad \begin{cases} Sn\left[u, \gamma\right] \cong \sin(u) \quad when \quad \gamma << 1 \\ y(0) = \dfrac{(u_3^2(0) - u_{3a}^2)}{(u_{3b}^2 - u_{3a}^2)} = \dfrac{u_3^2(0)}{u_2^2(0)} = 0 \\ \xi_0 = 0 \end{cases}$$

Inverting the normalized variables to obtain the field intensities:

$$\begin{cases} I_3(z) \cong \dfrac{\omega_3}{\omega_2}\dfrac{I_2(0)}{\cos^2\beta}\sin^2(\dfrac{z}{\ell_u}) \\ I_2(z) \cong I_2(0)\cos^2(\dfrac{z}{\ell_u}) \\ I_1(z) \cong I_1(0) \end{cases}$$

$$where \quad \frac{z}{\ell_u} = u_1(0)\xi = \sqrt{\frac{I_1(0)}{\omega_1 W}}\sqrt{\frac{8\pi^3 W\,\omega_1^2\omega_2^2\omega_3^2}{c^6 K_{1Z}K_{2Z}K_{3Z}\cos^2\beta}}\cdot\chi_{eff}^{(2)}\cdot z$$

2. Second – Harmonic Generation (SHG)

Initial condition: $\begin{cases} u_1^2(0) = u_2^2(0) \\ u_3(0) = 0 \quad \Rightarrow \quad \Gamma = 0 \end{cases}$

The denominator polynomial inside the radical of the integral

$$u_3^2(m_2 - u_3^2)(m_1 - u_3^2) - \Gamma^2 = u_3^2(m_1 - u_3^2)^2$$

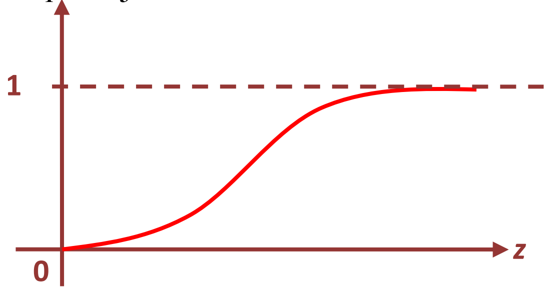
Therefore, the integral becomes:

$$\xi = \int_{u_3(0)}^{u_3(\xi)} \frac{du_3}{m_1 - u_3^2} = \frac{1}{u_1(0)}\tanh^{-1}\left[\frac{u_3(\xi)}{u_1(0)}\right]$$

Converting back to the beam intensities

$$\begin{aligned} I_3(z) &= (\frac{\omega_3}{\omega_1})\cdot\frac{I_1(0)}{\cos^2\beta}\tanh^2(\frac{z}{\ell_u}) \\ I_2(z) &= I_2(0)\operatorname{sec}h^2(\frac{z}{\ell_u}) \\ I_1(z) &= I_1(0)\operatorname{sec}h^2(\frac{z}{\ell_u}) \end{aligned}$$

$$\eta = \frac{I_3(z)\omega_1}{I_1(0)\omega_3}$$



With $\frac{I_2(0)}{I_1(0)} = \frac{\omega_2}{\omega_1}$ for equal photon number

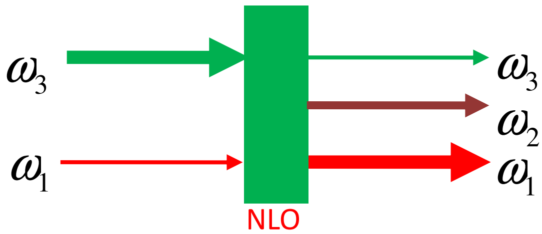
When $\omega_1 = \omega_2 = \omega$ and $\omega_3 = 2\omega$, then $I_1(0) = I_2(0) = \frac{1}{2} I_\omega(0)$

$$\begin{cases} I_{2\omega}(z) = \dfrac{I_\omega(0)}{\cos^2\beta} \tanh^2\left(\dfrac{z}{\ell_{SH}}\right) \\ I_\omega(z) = I_\omega(0) \operatorname{sec} h^2\left(\dfrac{z}{\ell_{SH}}\right) \end{cases}$$

$$\frac{1}{\ell_{SH}} = 4.7\times10^{-2}\sqrt{I_\omega\left(\frac{MW}{cm^2}\right)} \qquad with \;\; \chi_{eff}^{(2)} = 1.5\times10^{-9}\, esu$$

3. Optical Parametric Amplification (OPA)

Initial condition: $\begin{cases} u_2^2(0) = 0 \quad \Rightarrow \Gamma = 0 \\ u_1^2(0) << u_3^2(0) \end{cases}$



Three roots are determined to be

$u_{3a}^2 = 0,$

$u_{3b}^2 = m_2 = u_3^2(0)$

$u_{3c}^2 = m_1 = u_3^2(0) + u_1^2(0)$

$$y(0) = \frac{u_3^2(0) - u_{3a}^2}{u_{3b}^2 - u_{3a}^2} = 1$$

Note $F\left[\sin^{-1} y(0), \gamma\right] = F\left[\frac{\pi}{2}, \gamma\right] = K(\gamma)$

$$\xi_0 = \frac{F}{\sqrt{(u_{3c}^2 - u_{3a}^2)}} = \frac{K(\gamma)}{\sqrt{(u_{3c}^2 - u_{3a}^2)}}$$

and $\gamma^2 = \frac{u_3^2(0)}{[u_3^2(0) + u_1^2(0)]}$ = **photon number ratio of input beams**

The solution for the pump beam becomes

$$u_3^2(\xi) = u_{3a}^2 + (u_{3b}^2 - u_{3a}^2)Sn^2\left[\sqrt{u_{3c}^2 - u_{3a}^2}(\xi+\xi_0),\gamma\right]$$

$$= u_3^2(0)Sn^2\left[\sqrt{u_3^2(0)+u_1^2(0)}\left(\xi + \frac{K(\gamma)}{\sqrt{u_3^2(0)+u_1^2(0)}}\right),\gamma\right]$$

$$= u_3^2(0)Sn^2\left[\sqrt{u_3^2(0)+u_1^2(0)}\,\xi - K(\gamma),\gamma\right]$$

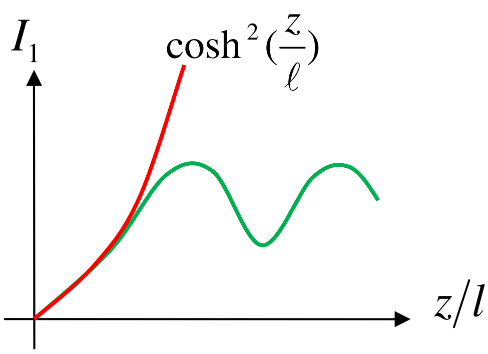
$$= u_3^2(0)Sn^2\left[\frac{1}{\gamma}u_3^2(0)(\xi-\xi_0),\gamma\right]$$

Let $\cos^2\beta = 1$ to neglect the walk-off effect

Then:

$$\begin{cases} I_1(z) = I_1(0) + \left(\frac{\omega_1}{\omega_3}\right)I_3(0)\left\{1 - Sn^2\left[\frac{(z-z_0)}{\ell},\gamma\right]\right\} \\ I_2(z) = \left(\frac{\omega_2}{\omega_1}\right)I_3(0)\left\{1 - Sn^2\left[\frac{(z-z_0)}{\ell},\gamma\right]\right\} \\ I_3(z) = I_3(0)Sn^2\left[\frac{(z-z_0)}{\ell},\gamma\right] \end{cases}$$

$$where \begin{cases} \frac{z}{\ell} = \frac{u_3(0)\xi}{\gamma} \\ \frac{z_0}{\ell} = \frac{u_3(0)\xi_0}{\gamma} \end{cases}$$



(i) When $|\xi| << 1$ for negligible pump depletion and

$$u_1^2,\ u_2^2 << u_3^2 \quad \Rightarrow \quad \gamma \to 1$$

Note: $\omega_1\left(\frac{I_3(0)}{\omega_3}\right) = I_1(0)\frac{\gamma^2}{1-\gamma^2}$

Therefore: $$\begin{cases} I_1(z) = I_1(0) + (\frac{\gamma^2}{1-\gamma^2}) I_1(0) \left\{ 1 - Sn^2 \left[\frac{(z-z_0)}{\ell}, \gamma \right] \right\} \\ I_2(z) = (\frac{\omega_2}{\omega_1})(\frac{\gamma^2}{1-\gamma^2}) I_1(0) \left\{ 1 - Sn^2 \left[\frac{(z-z_0)}{\ell}, \gamma \right] \right\} \\ I_3(z) = I_3(0) Sn^2 \left[\frac{(z-z_0)}{\ell}, \gamma \right] \end{cases}$$

Note: $$(\frac{\gamma^2}{1-\gamma^2}) \left\{ 1 - Sn^2 \left[\frac{(z-z_0)}{\ell}, \gamma \right] \right\} \xrightarrow[\gamma \to 1]{} \gamma^2 Sn^2 (\frac{z}{\ell}) \frac{1}{dn^2(\frac{z}{\ell})} = \sinh^2(\frac{z}{\ell})$$

$$\therefore \begin{cases} I_1(z) \cong I_1(0) \cosh^2(\frac{z}{\ell}) \\ I_2(z) \cong I_1(0) \frac{\omega_1}{\omega_2} \sinh^2(\frac{z}{\ell}) \\ I_3(z) \cong I_3(0) \end{cases}$$

(ii) When phase mismatch appears, $\Delta K \neq 0$

$$u_3^2 (m_2 - u_3^2)(m_1 - u_3^2) - (\frac{\Delta S}{2})^2 (m_2 - u_3^2)^2$$

$$\Rightarrow u_{3a}^2 = (\frac{\Delta S}{2})^2$$

$$u_{3b}^2 = m_2 = u_3^2(0)$$

$$u_{3c}^2 = u_3^2(0) + u_1^2(0)$$

$$\therefore \frac{z}{\ell} = \sqrt{\frac{u_3^2(0)\xi^2}{\gamma^2} - (\frac{\Delta K}{2})^2} \cdot z = \sqrt{\Gamma_0^2 - (\frac{\Delta K}{2})^2} \cdot z$$

$$\textit{where } \Gamma_0^2 = \frac{u_3^2(0)\xi^2}{\gamma^2}$$

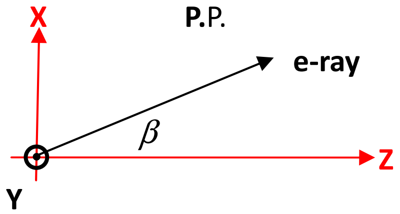
$$\therefore \begin{cases} I_1(z) \cong I_1(0) \cosh^2(\sqrt{\Gamma_0^2 - (\frac{\Delta K}{2})^2} \cdot z) \\ I_2(z) \cong I_1(0) \frac{\omega_2}{\omega_1} \sinh^2(\sqrt{\Gamma_0^2 - (\frac{\Delta K}{2})^2} \cdot z) \\ I_3(z) \cong I_3(0) \end{cases}$$

i.e., Parametric gain can be reduced by the phase mismatching.

Summaries of Nonlinear Frequency Conversion

$$\left(\nabla\times\nabla\times+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(\vec{r},t)+\frac{\varepsilon_0}{c^2}\frac{\partial^2 E}{\partial t^2}=-\frac{4\pi}{c^2}\frac{\partial^2 P_{NL}}{\partial t^2}$$
$$where\ \ P_{NL}(\vec{r},t)=\chi^{(2)}E^2(r,t)$$

Let $\vec{E}(r,t)=\frac{1}{2}\sum_{n=1}^{3}\left(\hat{e}_n A_n(r,t)e^{i(\omega_n t-K_n r)}+c.c.\right)$



By invoking **SVA**

$$\Rightarrow\begin{cases}\hat{M}_1A_1=i\xi_1A_3A_2^*e^{i\Delta KZ}\\ \hat{M}_2A_2=i\xi_2A_3A_1^*e^{i\Delta KZ}\\ \hat{M}_3A_3=i\xi_3A_1A_2e^{i\Delta KZ}\end{cases},$$

where

$$\hat{M}_n=\frac{\partial}{\partial Z}+\rho_n\frac{\partial}{\partial x}+\frac{i}{2K_n}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{u_n}\frac{\partial}{\partial t}+ig_n\frac{\partial^2}{\partial t^2}+\sigma_n$$
$$\xi_n=\frac{2\pi K_n}{n_n^2}(\hat{e}_{out}\cdot\chi^{(2)}:\hat{e}_{in2}\hat{e}_{in1})$$

$$\sigma_n: absorption\ loss=\frac{K_n}{2n_n^2}(\hat{e}\cdot I_m\varepsilon_0\cdot\hat{e}_n)$$

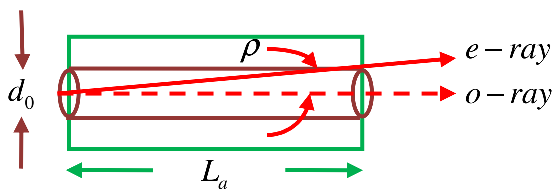
$$\rho_n: walk-off\ \ angle\ on\ the\ XZ\ plane$$

$$g_n: dispersion\ spreading=\frac{1}{2}\left(\frac{\partial^2 K_n}{\partial\omega^2}\right)_{\omega_n}$$

$$u_n: group\ velocity=\left(\frac{\partial\omega}{\partial K}\right)_{\omega_n}$$

Effective lengths for the interaction process

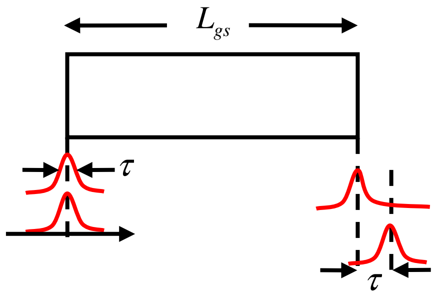
(1) **Aperture Length:** $L_a=\frac{d_0}{\rho},\ \ d_0=beam\ diameter$



(2) Quasi-static interaction length

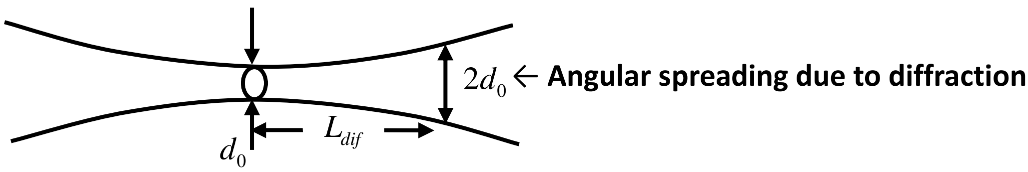
$$L_{gs} = \tau\left(\frac{1}{u_1} - \frac{1}{u_2}\right)^{-1} = \nu\tau$$

$$\nu = \left(\frac{1}{u_1} - \frac{1}{u_2}\right)^{-1}$$



(3) Diffraction Length

$$L_{dif} = kd_0^2$$



(4) Dispersion-Spreading Length

$$L_{ds} = \frac{\tau^2}{g} = \frac{2\tau^2}{\underbrace{\left(\frac{\partial^2 K_n}{\partial \omega^2}\right)_{\omega_n}}_{\text{material dispersion}}}$$

pulse broadening effect

τ **=pulse width**

(5) Nonlinear interaction length

$$L_{NL} = \frac{1}{\xi\sqrt{E_1^2(0) + E_2^2(0) + E_3^2(0)}}, \quad with\ \xi = \frac{2\pi\tilde{\omega}}{\mathrm{n}}\chi_{eff}^{(2)}$$

$L > L_{NL}$ **non-depleted pump beam assumption is incorrect**

Designing Process for NLO devices:

(a) Determine all the effective lengths of the process, compare them with the length of a nunlinear crystal, find out the effects that must be taken into account.

(b) Find the nonlinear interaction length, compare it with the crystal length, and determine whether the fixed-field approximation is valid or exact equations must be solved.

e.g., if $L > L_{NL}$, then non-depleted pump beam is not valid.

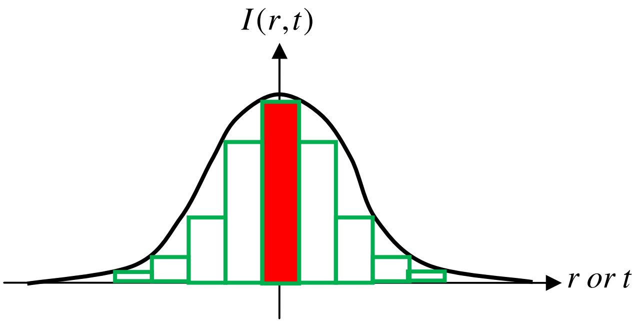
If the crystal length, L, is smaller than each effective length, then

$$\hat{M}_n \Rightarrow \frac{\partial}{\partial Z} + \sigma_n$$

The beam or pulse envelope of the radiation being converted is approximation by a step-wise function, the field amplitude inside of each step being constant.

For each step, the conversion efficiency is calculated by the equations for plane waves.

Then the results are summed with respect to the transverse coordinates (or time) and the power of the beam (or pulse) of the resulting radiation is determined.



Limitations of the High Conversion Efficiency

We can not increase the pump beam intensity by focusing the optical beam indefinitely. This is because there are actually some limits on the focused beam:

(a) Optical damage threshold of NLO materials

(b) Effective interaction length (beam walk off) ~ This may be partially relieved by using 90^o -noncritical phase matching scheme.

Poor beam quality produces an ill-effect on the nonlinear conversion efficiency

(a) Hot spot $\rightarrow$ reduced effective interaction length

- **inhomogeneous medium**

$$\left|\Delta K \cdot \frac{\ell}{2}\right| < \frac{\pi}{2} \Rightarrow \Delta n < \frac{\lambda}{4\ell} \cong 10^{-5} \quad for \;\; \lambda = 1\,\mu m \;\;\&\;\; \ell = 1\,cm$$

- **Temperature uniformity**

If $\frac{dn}{dT} \sim 5\times10^{-5}$, then $\Delta T \sim 0.5^o K \Rightarrow \Delta n \sim 2.5\times10^{-5}$