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15		**Chapter Six: Vector Calculus** (**Lecture Note: PPT - 969 kB**) I. Fluid Flow: 1.Geneal FLow and Curved Surfaces II. Vector Derivatives: Div, Curl, and Strain
16		III. Computing the Divergence and the Curl of a Vector Field IV. Gradient, Divergence, and Curl Operations
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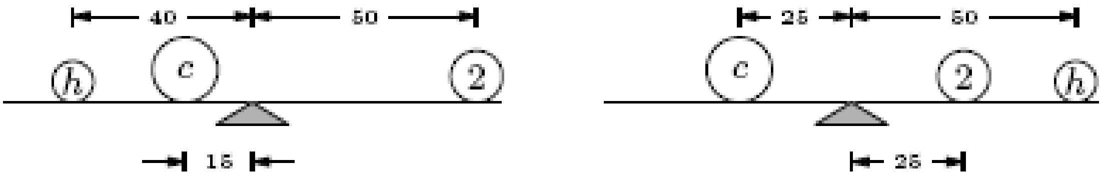
18		**Final Examination**
教科書 (請註明書名、作者、出版社、出版年等資訊)		•Gilber Strang: ***Introduction to Linear Algebra***, third edition (Wellsley-Cambridge Press, Box 812060 Wellesley MA 02482) (歐亞圖書公司) •Lecture notes have been posted whenever possible. Please see http://www.jyhuang.idv.tw

Chapter 1 *Linear Systems*

I. Solving Linear Systems

Systems of linear equations are common in science and mathematics.
To give you some ideas, consider an example that gives three objects, one with a mass known to be 2 *kg*, and are asked to find the two unknown masses.
Suppose that experimentation with a meter stick produces these two balances.



Since the sum of moments on the left of each balance equals the sum of moments on the right (the law of torque balance):

$$40h + 15c = 100$$
$$25c = 50 + 50h$$

Now you shall be able to find out the unique solution of the two unknown masses.

The second example is from Chemistry. Let us mix toluene C_7H_8 with nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ (TNT) along with the byproduct water:

$$x\,\mathrm{C_7H_8} \;+\; y\,\mathrm{HNO_3} \;\longrightarrow\; z\,\mathrm{C_7H_5O_6N_3} \;+\; w\,\mathrm{H_2O}$$

The number of atoms of each element present before the reaction must equal the number present afterward (the mass conservation law).

$$\begin{aligned} 7x &= 7z \\ 8x + 1y &= 5z + 2w \\ 1y &= 3z \\ 3y &= 6z + 1w \end{aligned}$$

However, you will soon find that unlike to the first problem the linear equation system of the second example possesses an infinite number of solutions.

I.1 Gauss' Method

1.1 Definition A *linear equation* in variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the equation's *coefficients* and $d \in \mathbb{R}$ is the *constant*. An n-tuple $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a *solution* of, or *satisfies*, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \ldots + a_ns_n = d$.

A *system of linear equations*

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= d_2 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= d_m \end{aligned}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that n-tuple is a solution of all of the equations in the system.

- **Ex : (Linear or Nonlinear)**

Linear $(a)\ 3x + 2y = 7$

$(b)\ \frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear $(c)\ x_1 - 2x_2 + 10x_3 + x_4 = 0$

$(d)\ (\sin\frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

Nonlinear $(e)\ xy + z = 2$ (not the first power)

$(f)\ e^x - 2y = 4$ Nonlinear (Exponential)

Nonlinear $(g)\ \sin x_1 + 2x_2 - 3x_3 = 0$ (trigonometric functions)

$(h)\ \frac{1}{x} + \frac{1}{y} = 4$ Nonlinear (not the first power)

Finding the set of all solutions is *solving* the system. There is an algorithm that always works. The next example introduces that algorithm, called ***Gauss' method***.

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \end{aligned}$$

$$\xrightarrow{\text{swap row 1 with row 3}} \begin{aligned} \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

$$\xrightarrow{\text{multiply row 1 by 3}} \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

$$\xrightarrow{\text{add } -1 \text{ times row 1 to row 2}} \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ -x_2 - 2x_3 &= -7 \\ 3x_3 &= 9 \end{aligned}$$

1.4 Theorem (Gauss' method) If a linear system is changed to another by one of these operations

(1) an equation is swapped with another

(2) an equation has both sides multiplied by a nonzero constant

(3) an equation is replaced by the sum of itself and a multiple of another

then the two systems have the same set of solutions.

Each of those three operations has a restriction. Multiplying a row by 0 is not allowed because obviously that can change the solution set of the system. Similarly, adding a multiple of a row to itself is not allowed because adding -1 times the row to itself has the effect of multiplying the row by 0. Finally, swapping a row with itself is disallowed.

The three ***elementary row operations*** from Theorem 1.4 are called ***swapping***, ***rescaling***, and ***pivoting***.

Proof

Consider this swap of row i with row j.

$$\begin{array}{rcl} a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n = d_1 & & a_{1,1}x_1 + a_{1,2}x_2 + \cdots a_{1,n}x_n = d_1 \\ \vdots & & \vdots \\ a_{i,1}x_1 + a_{i,2}x_2 + \cdots a_{i,n}x_n = d_i & & a_{j,1}x_1 + a_{j,2}x_2 + \cdots a_{j,n}x_n = d_j \\ \vdots & \longrightarrow & \vdots \\ a_{j,1}x_1 + a_{j,2}x_2 + \cdots a_{j,n}x_n = d_j & & a_{i,1}x_1 + a_{i,2}x_2 + \cdots a_{i,n}x_n = d_i \\ \vdots & & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots a_{m,n}x_n = d_m & & a_{m,1}x_1 + a_{m,2}x_2 + \cdots a_{m,n}x_n = d_m \end{array}$$

The n-tuple $(s_1, \ldots, s_n)$ satisfies the system before the swap if and only if substituting the values, the s's, for the variables, the x's, gives true statements:

$a_{1,1}s_1+a_{1,2}s_2+\cdots+a_{1,n}s_n = d_1$ **and** . . . $\boldsymbol{a_{i,1}s_1+a_{i,2}s_2+\cdots+a_{i,n}s_n = d_i}$ **and** . . .

$a_{j,1} s_1 + a_{j,2} s_2 + \cdots + a_{j,n} s_n = d_j$ **and** $\ldots$ $a_{m,1} s_1 + a_{m,2} s_2 + \cdots + a_{m,n} s_n = d_m$.

In a requirement consisting of statements **and-ed** together we can rearrange the order of the statements, so that this requirement is met if and only if

$a_{1,2}s_2 + \cdots + a_{1,n}s_n = d_1$ and $\ldots$ $a_{j,1}s_1 + a_{j,2}s_2 + \cdots + a_{j,n}s_n = d_j$ and $\ldots$ $a_{i,1}s_1 + a_{i,2}s_2 + \cdots + a_{i,n}s_n = d_i$ and $\ldots$ $a_{m,1}s_1 + a_{m,2}s_2 + \cdots + a_{m,n}s_n = d_m$.

This is exactly the requirement that $(s_1, \ldots, s_n)$ solves the system after the row swap.

Q.E.D.

When writing out the calculations, we will abbreviate '**row *i***' by 'ρ_i'. For instance, we will denote a pivot operation by $k\rho_i + \rho_j$, with the row that is changed written second.

$$\begin{array}{rcl} x + y = 0 \\ 2x - y + 3z = 3 \\ x - 2y - z = 3 \end{array} \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \begin{array}{r} x + y = 0 \\ -3y + 3z = 3 \\ -3y - z = 3 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_3} \begin{array}{r} x + y = 0 \\ -3y + 3z = 3 \\ -4z = 0 \end{array}$$

1.9 Definition In each row, the first variable with a nonzero coefficient is the row's *leading variable*. A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it (except for the leading variable in the first row).

- **Row-echelon form:** satisfies the conditions (1), (2), and (3)
- **Reduced row-echelon form:** (1), (2), (3), and (4).

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).

(3) For two successive nonzero rows, the leading 1 in the higher row is farther to the right than the leading 1 in the row above it.

(4) Every column that contains a leading 1 has zeros everywhere else.

■ Ex : **echelon form** or **reduced echelon form**

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ (row-echelon form)

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row-echelon form)

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ (row-echelon form)

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ (reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$ (non echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$ (non echelon form)

Linear systems need not have the same number of equations as unknowns.

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= -2 \end{aligned} \quad \xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2} \quad \begin{aligned} x + 3y &= 1 \\ -5y &= -5 \\ -4y &= -4 \end{aligned}$$

$$\xrightarrow{-(4/5)\rho_2+\rho_3} \quad \begin{aligned} x + 3y &= 1 \\ -5y &= -5 \\ 0 &= 0 \end{aligned}$$

Note: One of the equations is redundant.

Another way that linear systems can differ from the examples shown earlier is that some linear systems do not have a **unique solution**. This can happen in two ways:

1. No pair of numbers satisfies all of the equations simultaneously (no solution).

$$\begin{aligned} x + 3y &= 1 \\ 2x + y &= -3 \\ 2x + 2y &= 0 \end{aligned} \quad \xrightarrow[-2\rho_1+\rho_3]{-2\rho_1+\rho_2} \quad \begin{aligned} x + 3y &= 1 \\ -5y &= -5 \\ -4y &= -2 \end{aligned}$$

$$\xrightarrow{-(4/5)\rho_2+\rho_3} \quad \begin{aligned} x + 3y &= 1 \\ -5y &= -5 \\ 0 &= 2 \end{aligned}$$

2. A linear system can fail to have a unique solution is to have many solutions.

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 8 \end{aligned} \quad \xrightarrow{-2\rho_1+\rho_2} \quad \begin{aligned} x + y &= 4 \\ 0 &= 0 \end{aligned}$$

Gauss' method uses the three elementary row operations to set a system up for back substitution:

If any step shows a contradictory equation then we can stop with the conclusion that the system has **no solutions**.

If we reach echelon form without a contradictory equation, and each variable is a leading variable in its row, then the system has **a unique solution** and we find it by back substitution.

Finally, if we reach echelon form without a contradictory equation, and at least one variable is not a leading variable, then the system has **many solutions**.

I.2 Describing the Solution Set

A linear system with a unique solution has a solution set with one element. A linear system with no solution has a solution set that is **empty**.

Solution sets are a challenge to describe only when they contain many elements.

$$\begin{array}{rcrcl} 2x & & +z & = & 3 \\ x & -y & -z & = & 1 \\ 3x & -y & & = & 4 \end{array} \xrightarrow[-(3/2)\rho_1+\rho_3]{-(1/2)\rho_1+\rho_2} \begin{array}{rcrcl} 2x & + & z & = & 3 \\ & -y & -(3/2)z & = & -1/2 \\ & -y & -(3/2)z & = & -1/2 \end{array}$$

$$\xrightarrow{-\rho_2+\rho_3} \begin{array}{rcrcl} 2x & + & z & = & 3 \\ & -y & -(3/2)z & = & -1/2 \\ & & 0 & = & 0 \end{array}$$

Here not all of the variables are leading variables. The solution set $\{(x, y, z) \mid 2x + z = 3 \text{ and } x - y - z = 1 \text{ and } 3x - y = 4\}$. We take the variable that does not lead any equation, z, and use it to describe the variables that do lead (*i.e.*, x and y). The solution set can be described as $\{(x, y, z) = ((3/2)-(1/2)z, (1/2)-(3/2)z, z) \mid z \in R\}$.

2.2 Definition The non-leading variables in an echelon-form linear system are *free variables*.

$$\begin{array}{rrrrcr} x+ & y+ & z- & w= & 1 \\ & y- & z+ & w= & -1 \\ 3x & & +6z & -6w= & 6 \\ & -y+ & z- & w= & 1 \end{array} \xrightarrow{-3\rho_1+\rho_3} \begin{array}{rrrrcr} x+ & y+ & z- & w= & 1 \\ & y- & z+ & w= & -1 \\ & -3y+ & 3z & -3w= & 3 \\ & -y+ & z- & w= & 1 \end{array}$$

$$\xrightarrow[\rho_2+\rho_4]{3\rho_2+\rho_3} \begin{array}{rcl} x+y+z-w & = & 1 \\ y-z+w & = & -1 \\ 0 & = & 0 \\ 0 & = & 0 \end{array}$$

The linear equation system ends with x and y leading, and with both z and w free.

The solution set is
$\{2-2z+2w, -1+z-w, z, w) \mid z, w \in \mathbb{R}\}$.
We refer to a variable used to describe a family of solutions as a ***parameter*** and we say that the set above is *parametrized* with z and w.

2.6 Definition An $m{\times}n$ *matrix* is a rectangular array of numbers with m *rows* and n *columns*. Each number in the matrix is an *entry*,

We shall use $M_{n\times m}$ to denote the collection of $n{\times}m$ matrices. We can abbreviate this linear system

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ x_2 - x_3 &= 0 \\ x_1 + 2x_3 &= 4 \end{aligned}$$ with this matrix $$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right).$$

In this notation, Gauss' method goes this way:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right) \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow{2\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the solution set is $\{(x = 4-2z, y = z, z) \mid z \in R\}$.

We will also use the array notation to clarify the descriptions of solution sets. A description like $\{(2-2z+2w, -1+z-w, z, w) \mid z, w \in \mathbb{R}\}$.

We will write them vertically, in one-column wide matrices.

$$\{\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid z, w \in \mathbb{R}\}.$$

2.8 Definition A *vector* (or *column vector*) is a matrix with a single column. A matrix with a single row is a *row vector*. The entries of a vector are its *components*.

2.9 Definition The linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ with unknowns $x_1, \ldots, x_n$ is *satisfied* by

$$\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$$

if $a_1s_1 + a_2s_2 + \cdots + a_ns_n = d$. A vector satisfies a linear system if it satisfies each equation in the system.

2.10 Definition The *vector sum* of $\vec{u}$ and $\vec{v}$ is this.

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

In general, two matrices with the same number of rows and the same number of columns add in this way, entry-by-entry.

2.11 Definition The *scalar multiplication* of the real number r and the vector $\vec{v}$ is this.

$$r \cdot \vec{v} = r \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix}$$

In general, any matrix is multiplied by a real number in this entry-by-entry way.

The linear system reduces with Gauss' method by

$$\begin{array}{rcl} 2x + y \quad - w &=& 4 \\ y \quad + w + u &=& 4 \\ x \quad - z + 2w &=& 0 \end{array}$$

$$\left(\begin{array}{cccccc|c} 2 & 1 & 0 & -1 & 0 & & 4 \\ 0 & 1 & 0 & 1 & 1 & & 4 \\ 1 & 0 & -1 & 2 & 0 & & 0 \end{array}\right) \xrightarrow{-(1/2)\rho_1+\rho_3} \left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & -1/2 & -1 & 5/2 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{(1/2)\rho_2+\rho_3} \left(\begin{array}{ccccc|c} 2 & 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & 3 & 1/2 & 0 \end{array}\right)$$

.

The solution set is

$\{(w + (1/2)u, 4 - w - u, 3w + (1/2)u, w, u) \mid w, u \in \mathbb{R}\}$, which can be written in vector form of

$$\{\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} w + \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} u \mid w, u \in \mathbb{R}\}$$

If we do Gauss' method in two different ways, must we get the same number of free variables?

If any two solution set descriptions have the same number of parameters, must those be the same variables?

Can we always describe solution sets as above, with a particular solution vector added to an unrestricted linear combination of some other vectors, *i.e.*,

$$\{\underbrace{\begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{w\begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 1/2 \\ -1 \\ 1/2 \\ 0 \\ 1 \end{pmatrix}}_{\text{unrestricted combination}} \mid w, u \in \mathbb{R}\}$$

I.3 General = Particular + Homogeneous

Infinite Solution Set: An infinite solution set exhibits the pattern that a vector that is a particular solution of the system added to an unrestricted combination of some other vectors (a non-$\vec{0}$ solution).

Unique Solution: A one-element solution set has a particular solution, and the unrestricted combination part is $\vec{0}$ (a trivial sum of no vectors).

No Solution: A zero-element solution set fits the pattern since there is no particular solution, and so the set of sums of that form is empty.

3.2 Definition A linear equation is *homogeneous* if it has a constant of zero, that is, if it can be put in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

$$\begin{array}{l} 3x + 4y = 3 \\ 2x - \ \ y = 1 \end{array} \xrightarrow{-(2/3)\rho_1+\rho_2} \begin{array}{l} 3x + \qquad\quad 4y = 3 \\ \qquad -(11/3)y = -1 \end{array}$$

$$\begin{array}{l} 3x + 4y = 0 \\ 2x - \ \ y = 0 \end{array} \xrightarrow{-(2/3)\rho_1+\rho_2} \begin{array}{l} 3x + \qquad\quad 4y = 0 \\ \qquad -(11/3)y = 0 \end{array}$$

Studying the associated homogeneous system has a great advantage over studying the original system. Non-homogeneous systems can be inconsistent (*i.e.*, no solution). But a homogeneous system must be consistent since there is always at least one solution, the vector of zeros (**zero vector**).

Some homogeneous systems have many solutions.

$$\begin{array}{rcrcrcrcl} 7x & & & - & 7z & & & = & 0 \\ 8x & + & y & - & 5z & - & 2k & = & 0 \\ & & y & - & 3z & & & = & 0 \\ & & 3y & - & 6z & - & k & = & 0 \end{array} \xrightarrow{-(8/7)\rho_1+\rho_2} \begin{array}{rcrcrcrcl} 7x & & & - & 7z & & & = & 0 \\ & & y & + & 3z & - & 2w & = & 0 \\ & & y & - & 3z & & & = & 0 \\ & & 3y & - & 6z & - & w & = & 0 \end{array}$$

$$\xrightarrow[-3\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{array}{rcrcrcrcl} 7x & & & - & 7z & & & = & 0 \\ & & y & + & 3z & - & 2w & = & 0 \\ & & & & -6z & + & 2w & = & 0 \\ & & & & -15z & + & 5w & = & 0 \end{array}$$

$$\xrightarrow{-(5/2)\rho_3+\rho_4} \begin{array}{rcrcrcrcl} 7x & & & - & 7z & & & = & 0 \\ & & y & + & 3z & - & 2w & = & 0 \\ & & & & -6z & + & 2w & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

The solution set:

$$\{\begin{pmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{pmatrix} w \mid w \in \mathbb{R}\}.$$

We had encountered a linear system of

$$\begin{array}{rcrcrcrcrcl} 2x & + & y & & & - & w & & & = & 4 \\ & & y & & & + & w & + & u & = & 4 \\ x & & & - & z & + & 2w & & & = & 0 \end{array}$$

, whose solution set can be expressed as

$$\{\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} w + \begin{pmatrix} 1/2 \\ -1 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} u \mid w, u \in \mathbb{R}\}.$$

3.7 Lemma For any homogeneous linear system there exist vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ such that the solution set of the system is

$$\{c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where k is the number of free variables in an echelon form version of the system.

3.8 Lemma For a linear system, where $\vec{p}$ is any particular solution, the solution set equals this set.

$$\{\vec{p} + \vec{h} \mid \vec{h} \text{ satisfies the associated homogeneous system}\}$$

Proof

We will show that [1] any solution to the system belongs to the above set and that
[2] anything in the set is a solution to the linear system.

[1] If a vector solves the system then it is in the set described above. Assume that $\vec{s}$ solves the system. Then $\vec{s} - \vec{p}$ solves the associated homogeneous system since for each equation index *i*,

$$\begin{aligned} a_{i,1}(s_1 - p_1) + \cdots + a_{i,n}(s_n - p_n) &= (a_{i,1}s_1 + \cdots + a_{i,n}s_n) \\ &\quad - (a_{i,1}p_1 + \cdots + a_{i,n}p_n) \\ &= d_i - d_i \\ &= 0 \end{aligned}$$

where p_j and s_j are the *j*-th components of $\vec{p}$ and $\vec{s}$.

We can then write $\vec{s} - \vec{p}$ as $\vec{h}$, where $\vec{h}$ solves the associated homogeneous system, to express $\vec{s}$ in the required $\vec{p} + \vec{h}$ form.

[2] For a vector with the form $\vec{p} + \vec{h}$, where $\vec{p}$ solves the linear system and $\vec{h}$ solves the associated homogeneous linear system, and note that for any equation index *i*,

$$\begin{aligned} a_{i,1}(p_1 + h_1) + \cdots + a_{i,n}(p_n + h_n) &= (a_{i,1}p_1 + \cdots + a_{i,n}p_n) \\ &\quad + (a_{i,1}h_1 + \cdots + a_{i,n}h_n) \\ &= d_i + 0 \\ &= d_i \end{aligned}$$

The vector then solves the given linear system.

3.1 Theorem For any linear system there are vectors $\vec{\beta}_1, \ldots, \vec{\beta}_k$ such that the solution set can be described as

$$\{\vec{p} + c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

where $\vec{p}$ is any particular solution, and where the system has k free variables.

3.11 Corollary Solution sets of linear systems are either empty, have one element, or have infinitely many elements.

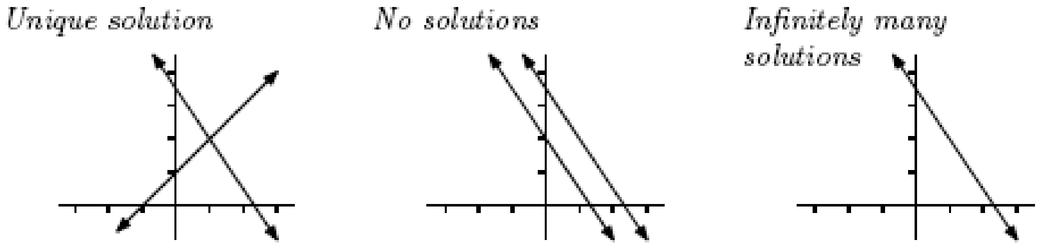
Now, apply Lemma 3.8 to conclude that a solution set $\{\vec{p} + \vec{h} \mid \vec{h} \text{ solves the associated homogeneous system}\}$ is either empty (if there is no particular solution $\vec{p}$), or has one element (if there is a $\vec{p}$ and the homogeneous system has the unique solution $\vec{0}$), or is infinite (if there is a $\vec{p}$ and the homogeneous system has a non-$\vec{0}$ solution, and thus by the prior paragraph has infinitely many solutions).

number of solutions of the associated homogeneous system

		one	*infinitely many*
particular solution exists?	*yes*	unique solution	infinitely many solutions
	no	no solutions	no solutions

3.12 Definition A square matrix is *nonsingular* if it is the matrix of coefficients of a homogeneous system with a unique solution. It is *singular* otherwise, that is, if it is the matrix of coefficients of a homogeneous system with infinitely many solutions.

II Linear Geometry of *n*-Space



$3x + 2y = 7$ $x - y = -1$	$3x + 2y = 7$ $3x + 2y = 4$	$3x + 2y = 7$ $6x + 4y = 14$

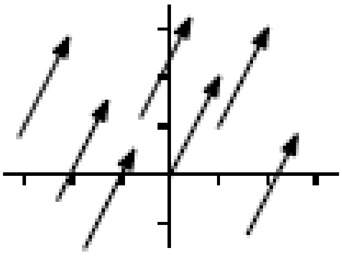
There are only three types of solution sets—singleton, empty, and infinite. In the special case of systems with two equations and two unknowns this is easy to see. Draw each two-unknowns equation as a line in the plane and then the two lines could have a unique intersection, be parallel, or be the same line.

II.1 Vectors in Space

An object comprised of a magnitude and a direction is a *vector* ('vector' is Latin for "carrier" or "traveler"). We can draw a vector as having some length, and pointing somewhere.

A vector starts at (a_1, a_2), extend to (b_1, b_2) can be described as $\begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

The following free vectors are equal because they have equal lengths and equal directions.



We often draw the arrow as starting at the origin, and we then say it is in the **canonical position** (or *natural position*).

For the vector starts at (a_1, a_2), extend to (b_1, b_2), in its canonical position then it starts at the origin and extends to the endpoint $(b_1 - a_1, b_2 - a_2)$.

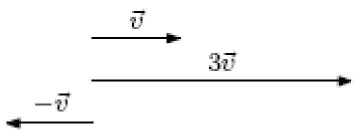
We will call both of the sets in R^2, as row vector and column vector

$$\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\} \qquad \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}.$$

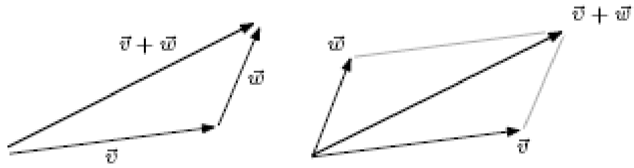
We can define some vector operations with an algebraic approach

$$r \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} rv_1 \\ rv_2 \end{pmatrix} \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}.$$

We can interpret those operations geometrically. For instance, if $\vec{v}$ represents a displacement then $3\vec{v}$ represents a displacement in the same direction but three times as far.



And, where $\vec{v}$ and $\vec{w}$ represent displacements, $\vec{v}+\vec{w}$ represents those displacements combined.



The above drawings show how vectors and vector operations behave in R^2. We can extend to R^3, or to even higher-dimensional spaces,

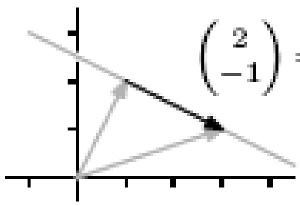
$$\mathbb{R}^n = \{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

with the obvious generalization: the free vector that, if it starts at $(a_1, \ldots, a_n)$, ends at $(b_1, \ldots, b_n)$, is represented by this column

$$\begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}.$$

In $\mathbf{R}^2$, the line through (1, 2) and (3, 1) is comprised of (the endpoints of) the vectors in this set

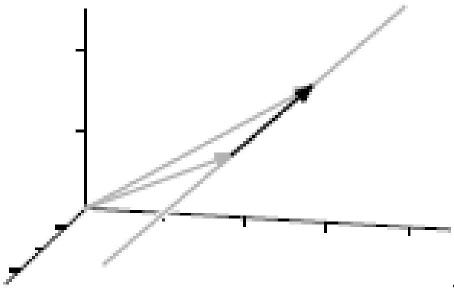
$$\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \}$$



$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Similarly in R^3, the line through (1, 2, 1) and (2, 3, 2) is the set of vectors of this form

$$\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \}$$

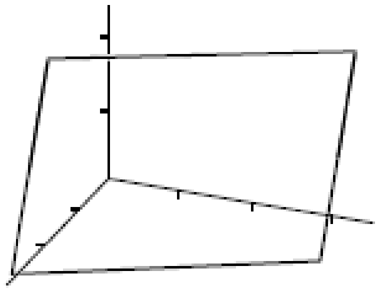


A line uses one parameter, so that there is freedom to move back and forth in one dimension. Therefore, a plane must involve two. For example, the plane through the points (1, 0, 5), (2, 1, −3), and (−2, 4, 0.5) consists of (endpoints of) the vectors in

$$\{\begin{pmatrix}1\\0\\5\end{pmatrix}+t\cdot\begin{pmatrix}1\\1\\-8\end{pmatrix}+s\cdot\begin{pmatrix}-3\\4\\-4.5\end{pmatrix}\mid t,s\in\mathbb{R}\}$$

.

A description of planes that is often encountered in algebra and calculus uses a single equation as the condition that describes the relationship among the first, second, and third coordinates of points in a plane.



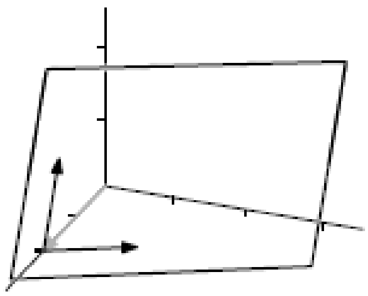
$$P=\{\begin{pmatrix}x\\y\\z\end{pmatrix}\mid 2x+y+z=4\}$$

.

The translation from such a description to the vector description is to think of the condition as a one-equation linear system and parametrize $x = (1/2)(4 - y - z)$ to yield

$$P=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2-0.5y-0.5z\\y\\z\end{pmatrix}\middle|\, y,z\in R\right\}.$$

Therefore,



$$P=\{\begin{pmatrix}2\\0\\0\end{pmatrix}+\begin{pmatrix}-0.5\\1\\0\end{pmatrix}y+\begin{pmatrix}-0.5\\0\\1\end{pmatrix}z\mid y,z\in\mathbb{R}\}$$

.

Generalizing from lines and planes, we define a *k-dimensional linear surface* (or *k-flat*) in R^n to be

$$\{\vec{p}+t_1\vec{v}_1+t_2\vec{v}_2+....+t_k\vec{v}_k \mid t_1,t_2,...,t_k\in R\}$$

where $\vec{v}_1,\vec{v}_2....,\vec{v}_k\in R^n$.

We finish this subsection by restating our conclusions from the first section in the following geometric terms:

First, **the solution set of a linear system with *n* unknowns is a linear surface in R^n**. Specifically, it is a ***k*-dimensional linear surface**, where ***k* is the number of free variables** in an echelon form version of the system.

Second, **the solution set of a homogeneous linear system is a linear surface passing through the origin**.

Finally, we can view **the general solution set of any linear system as being the solution set of its associated homogeneous system offset from the origin by a vector**, namely by any particular solution.

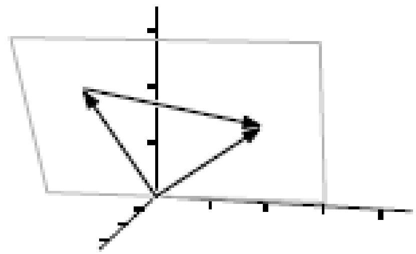
II.2 Length and Angle Measures

We'll follow an approach that a result familiar from R^2 and R^3, when generalized to arbitrary R^k, supports the idea that a line is straight and a plane is flat. Specifically, we'll see how to **do Euclidean geometry in a "plane" by giving a definition of the angle between two R^n vectors in the plane that they generate**.

2.1 Definition The *length* of a vector $\vec{v} \in \mathbb{R}^n$ is this.

$$\|\vec{v}\,\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

Consider three vectors $\vec{u}$, $\vec{v}$, and $\vec{u} - \vec{v}$, in **canonical position** and in the plane that they determine


,

apply the Law of Cosines, $|\vec{u} - \vec{v}\,|^2 = |\vec{u}\,|^2 + |\vec{v}\,|^2 - 2|\vec{u}\,||\vec{v}\,|\cos\theta$, where θ is the angle between the vectors. Expand both sides

$$(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 = (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2\,\|\vec{u}\,\|\,\|\vec{v}\,\|\cos\theta$$

and simplify

$$\theta = \arccos\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{\|\vec{u}\,\|\,\|\vec{v}\,\|}\right)$$
.

2.3 Definition The *dot product* (or *inner product*, or *scalar product*) of two n-component real vectors is the linear combination of their components.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

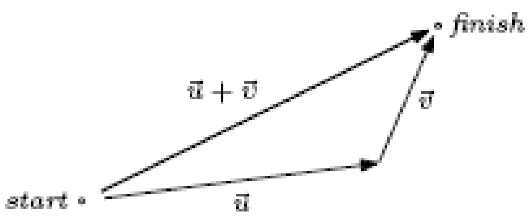
The dot product of a vector from R^n with a vector from R^m is defined only when n equals m. Note also this relationship between dot product, which is a real number, and the length.

2.5 Theorem (Triangle Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This inequality is the familiar saying, "The shortest distance between two points is in a straight line."



Proof

The desired inequality holds if and only if its square holds.

$$\|\vec{u} + \vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2$$
$$(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2$$
$$\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \leq \vec{u} \cdot \vec{u} + 2\|\vec{u}\|\,\|\vec{v}\| + \vec{v} \cdot \vec{v}$$
$$2\,\vec{u} \cdot \vec{v} \leq 2\|\vec{u}\|\,\|\vec{v}\|$$

That, in turn, holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $|\vec{u}|$ and $|\vec{v}|$

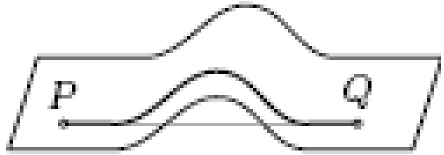
$$2(\|\vec{v}\|\,\vec{u}) \cdot (\|\vec{u}\|\,\vec{v}) \leq 2\|\vec{u}\|^2\,\|\vec{v}\|^2,$$

and rewriting

$$0 \leq \|\vec{u}\|^2\,\|\vec{v}\|^2 - 2(\|\vec{v}\|\,\vec{u}) \cdot (\|\vec{u}\|\,\vec{v}) + \|\vec{u}\|^2\,\|\vec{v}\|^2$$ is true.

Factoring $0 \leq (\|\vec{u}\|\,\vec{v} - \|\vec{v}\|\,\vec{u}) \cdot (\|\vec{u}\|\,\vec{v} - \|\vec{v}\|\,\vec{u})$ shows the inequality always true since it only says that the square of the length of the vector is not negative.

This result supports the intuition that even in higher-dimensional spaces, lines are straight and planes are flat. For any two points in a linear surface, the line segment connecting them is contained in that surface.



2.6 Corollary (Cauchy-Schwartz Inequality) For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\,\vec{u} \cdot \vec{v}\,| \le \|\,\vec{u}\,\|\;\|\vec{v}\,\|$$

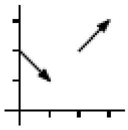
with equality if and only if one vector is a scalar multiple of the other.

2.7 Definition The *angle* between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\,\|\;\|\vec{v}\,\|} \right)$$

(the angle between the zero vector and any other vector is defined to be a right angle).

2.8 Example These vectors are orthogonal.



$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

III Reduced Echelon Form

Gauss' method can be done in more than one way. One example is that we sometimes have to swap rows and there can be more than one row to choose from.
Another example is:

$\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix}$ can be reduced to

(i) $\xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}$ or

(ii) $\xrightarrow{(1/2)\rho_1}$ then $\xrightarrow{-4\rho_1+\rho_2} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

(iii) $\xrightarrow{-2\rho_1+\rho_2}$ then $\xrightarrow{2\rho_2+\rho_1} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

The fact that the echelon form outcome of Gauss' method is not unique leaves us with some questions. Will any two echelon form versions of a linear system have the same number of free variables? Will they in fact have exactly the same variables free?

III.1 Gauss-Jordan Reduction

This approach is an extension of Gauss' method that has some advantages. This extension is called *Gauss-Jordan reduction*. It goes past echelon form to a more refined matrix form.

$$\begin{array}{rcrcrcr} x & + & y & - & 2z & = & -2 \\ & & y & + & 3z & = & 7 \\ x & & & - & z & = & -1 \end{array} \xrightarrow{-\rho_1+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\rho_2+\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{array}\right).$$

We can keep going to next step

$$\xrightarrow{(1/4)\rho_3} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow[2\rho_3+\rho_1]{-3\rho_3+\rho_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-\rho_2+\rho_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

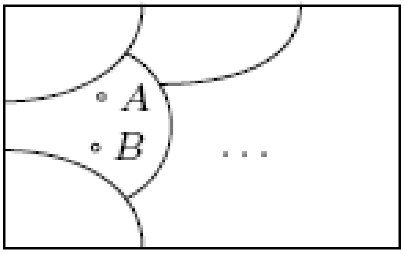
1.3 Definition A matrix is in *reduced echelon form* if, in addition to being in echelon form, each leading entry is a one and is the only nonzero entry in its column.

In any echelon form, plain or reduced, we can read off (1) when a linear system has an empty solution set because ***there is a contradictory equation***, we can read off (2) when a system has a one-element solution set because ***there is no contradiction and every variable is the leading variable*** in some row, and we can read off (3) when a system has an infinite solution set because ***there is no contradiction and at least one variable is free***.

Although we can start with the same linear system and proceed with row operations in many different ways, **the same reduced echelon form and the same parametrization** (using the unmodified free variables) **can be yielded**, *i.e.*, **the reduced echelon form version of a matrix is unique**.

1.6 Definition Two matrices that are interreducible by the elementary row operations are *row equivalent*.

The diagram below shows **the collection of all matrices as a box**. Inside that box, each matrix lies in some class. **Matrices are in the same class if and only if they are interreducible**. **The classes are disjoint since no matrix is in two distinct classes**. The collection of matrices has been partitioned into ***row equivalence classes***.



Every equivalence class contains one and only one reduced echelon form matrix. So each reduced echelon form matrix serves as **a representative** of its class.

2.1 Definition A *linear combination* of $x_1, \ldots, x_m$ is an expression of the form $c_1x_1 + c_2x_2 + \cdots + c_mx_m$ where the c's are scalars.

Corollary **Where one matrix reduces to another, each row of the second is a linear combination of the rows of the first.**

$$A = \begin{pmatrix} \cdots & \alpha_1 & \cdots \\ \cdots & \alpha_2 & \cdots \\ & \vdots & \\ \cdots & \alpha_m & \cdots \end{pmatrix} \qquad B = \begin{pmatrix} \cdots & \beta_1 & \cdots \\ \cdots & \beta_2 & \cdots \\ & \vdots & \\ \cdots & \beta_m & \cdots \end{pmatrix}$$

Proof.

In the base step:

that zero row reduction operations suffice, the two matrices are equal and each row of B is obviously a combination of A's rows:

$$\vec{\beta}_i = 0\cdot\vec{\alpha}_1 + 0\cdot\vec{\alpha}_2 + \ldots + 1\cdot\vec{\alpha}_i + \ldots + 0\cdot\vec{\alpha}_m .$$

For the inductive step:

[Given] with $t \geq 0$, if a matrix G can be derived from A in t or fewer operations then its rows are linear combinations of the A's rows.

[Prove] Consider a B that takes $t+1$ operations. Because there are more than zero operations, there must be a next-to-last matrix G so that $A \rightarrow \ldots \rightarrow G \rightarrow B$. This G is only t operations away from A and so the inductive hypothesis applies to it, that is, each row of G is a linear combination of the rows of A.

If the last operation, the one from G to B, is a **row swap** then the rows of B are just the rows of G reordered and thus each row of B is also a linear combination of the rows of A. The other two possibilities (**row rescaling**, **row pivoting**) for this last operation, that it multiplies a row by a scalar and that it adds a multiple of one row to another, both result in the rows of B being linear combinations of the rows of G. Therefore, each row of B is a linear combination of the rows of A.

With that, we have both the base step and the inductive step, and so the proposition follows.

A **D** **G** **B**

$$\begin{pmatrix}0 & 2\\ 1 & 1\end{pmatrix} \xrightarrow{\rho_1\leftrightarrow\rho_2} \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix} \xrightarrow{(1/2)\rho_2} \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \xrightarrow{-\rho_2+\rho_1} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\delta_1 = 0\cdot\alpha_1 + 1\cdot\alpha_2 \qquad \gamma_1 = 0\cdot\alpha_1 + 1\cdot\alpha_2 \qquad \beta_1 = (-1/2)\alpha_1 + 1\cdot\alpha_2$$
$$\delta_2 = 1\cdot\alpha_1 + 0\cdot\alpha_2 \qquad \gamma_2 = (1/2)\alpha_1 + 0\cdot\alpha_2 \qquad \beta_2 = (1/2)\alpha_1 + 0\cdot\alpha_2$$

This example gives us the insight that Gauss' method works by taking linear combinations of the rows. But why do we go to echelon form as a particularly simple version of a linear system? The answer is that echelon form is suitable for back substitution, because we have isolated the variables (*i.e.*, **the rows become mutually linear independent**).

Lemma In an echelon form matrix, no nonzero row is a linear combination of the other rows.

2.7 Theorem Each matrix is row equivalent to a unique reduced echelon form matrix.

There is one and only one reduced echelon form matrix in each row equivalence class. So the reduced echelon form is a *canonical form* for row equivalence: **the reduced echelon form matrices are representatives of the classes**.

Example

$\begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$ and $\begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix}$ are not equivalent since their corresponding reduced echelon form matrices $\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are not equal.