Linear Algebra and Vector Calculus

Final Examination

June 20, 2008

1. **(10%)** Find a basis and the dimension of the column space of the matrix $\boldsymbol{A}$

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

2. **(20%)** Find the least squares solution of the following system

$$\boldsymbol{A}\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{3} \end{bmatrix}$$

Find also the orthogonal projection of **b** on the column space of $\boldsymbol{A}$.

3. **(20%)** Let $\boldsymbol{\Pi : R^3(B) \to R^2(D)}$ be a projection onto the *xy*-plane. Give the matrix representation of this projection map with respect to the following bases:

$$B(R^3) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ and } D(R^2) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

4. **(25%)** Is $\vec{F}(x,y,z) = (2xy - z^3)\hat{i} + x^2\hat{j} - (3xz^2 + 1)\hat{k}$ conservative? If yes, find a scalar potential $\phi(x,y,z)$ such that $\vec{F}(x,y,z) = -\nabla\phi(x,y,z)$. How much work is done by this force in moving an object from the origin (0,0,0) to the point (1,1,1) along the path C: $y = x^2, z = x^3$. Verify the work done is equal to the difference of the potential energies at the two end positions (0,0,0) and (1,1,1) of the path $\boldsymbol{C}$..

5. **(25%)** Suppose that a satellite in space is taking photographs of Jupiter to be sent back to earth. The satellite digitizes the picture by subdividing it into tiny squares called *pixels*. Each pixel is represented by a single number that records the average light intensity in that square. If each photograph were divided into 500 × 500 pixels, it would have to send 250, 000 numbers to earth for each picture. This would take a great deal of time and would limit the number of photographs that could be transmitted. It is possible to approximate this matrix with a 'simpler' matrix which requires less storage."
For simplicity, let us consider a "picture" of only 3×2 pixels. The gray level specification for each square is determined by the corresponding entry in the matrix A

by $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. Approximate the picture with a simpler matrix which requires less storage by using singular value decomposition (SVD): $\boldsymbol{A} = \boldsymbol{U\Sigma V}^{T} = \sum_{i=1}^{rank(A)} \boldsymbol{\sigma}_i \boldsymbol{u}_i \boldsymbol{v}_i^T$,

where u and v are the eigenvectors of $\boldsymbol{AA}^T$ and $\boldsymbol{A}^T\boldsymbol{A}$, respectively, with

$\boldsymbol{AA}^T \boldsymbol{u}_i = \boldsymbol{\sigma}_i^2 \boldsymbol{u}_i$ and $\boldsymbol{A}^T \boldsymbol{A} \boldsymbol{v}_i = \boldsymbol{\sigma}_i^2 \boldsymbol{v}_i$.

Solutions:

1

$$A^T = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-3} & \mathbf{3} & \mathbf{2} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{4} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{6} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{3} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{-2} \end{bmatrix} \xrightarrow{G.E.} B = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-3} & \mathbf{3} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{9} & \mathbf{-5} & \mathbf{-6} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{matrix} \mathbf{w_1} \\ \mathbf{w_2} \\ \mathbf{w_3} \\ \\ \end{matrix}$$

Since $CS(A)=RS(A^{\mathrm{T}})$, a basis for $CS(A)$= a basis for $RS(A^{\mathrm{T}})$= {the nonzero row vectors of B} = $\{w_1, w_2, w_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$ is a basis for the column space of A. Therefore the dimension of

CS(A) is three, and the rank of A is three.

2. From $A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

the associated normal system $A^T A\mathbf{x} = A^T\mathbf{b} \quad \Rightarrow \quad \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

The least squares solution of Ax = b is

$$\mathbf{x}_{LS} = \begin{bmatrix} -\frac{5}{3} \\ \frac{3}{2} \end{bmatrix}$$

The orthogonal projection of **b** on the column space of A

$$\text{proj}_{CS(A)}\mathbf{b} = A\mathbf{x}_{LS} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-5}{3} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{6} \\ \frac{8}{6} \\ \frac{17}{6} \end{bmatrix}$$

3. To give a matrix representing this projection map, we first fix bases.

$$B(R^3)=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\} \text{ and } D(R^2)=\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

For each vector in the domain's basis, we find its image under the map.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}1\\0\end{pmatrix} \quad \begin{pmatrix}1\\1\\0\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}1\\1\end{pmatrix} \quad \begin{pmatrix}-1\\0\\1\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}-1\\0\end{pmatrix}$$

Then we find the representation of each image with respect to the codomain's basis

$$\mathrm{Rep}_D(\begin{pmatrix}1\\0\end{pmatrix})=\begin{pmatrix}1\\-1\end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix}1\\1\end{pmatrix})=\begin{pmatrix}0\\1\end{pmatrix} \quad \mathrm{Rep}_D(\begin{pmatrix}-1\\0\end{pmatrix})=\begin{pmatrix}-1\\1\end{pmatrix}$$

Finally, adjoining these representations gives the matrix representing Π with respect to $\boldsymbol{B}, \boldsymbol{D}$ is

$$\mathrm{Rep}_{B,D}(\pi)=\begin{pmatrix}1 & 0 & -1\\-1 & 1 & 1\end{pmatrix}_{B,D}$$

4. Note: $\nabla\times\vec{F}=0$, so we can find a scalar function $\phi=-x^2y+xz^3+z+C$, where $\boldsymbol{C}$ is a constant, such that $\vec{F}(x,y,z)=-\nabla\phi(x,y,z)$.

   We express the path parametrically by $\vec{X}=x\hat{i}+x^2\hat{j}+x^3\hat{k}$, $0\le x\le 1$, so that

   $d\vec{X}/dx=\hat{i}+2x\hat{j}+3x^2\hat{k}$. The force along the path is, in terms of the parameter x:

   $\vec{F}=(2x^3-x^9)\hat{i}+x^2\hat{j}-(3x^7+1)\hat{k}$. Then, the work done by this force is

   $$\int_C \vec{F}\cdot d\vec{X}=\int_0^1(-3x^2+4x^3-10x^9)dx=-1 \text{ and } \Delta\phi=\phi(1,1,1)-\phi(0,0,0)=1.$$

5. It is simple to get the answer! See my lecture note on SVD.