Linear Algebra and Vector Calculus
Final Examination

June 28, 2007

1. (15%) Let $B = \{1, x, e^x, xe^x\}$ be a basis of a subspace $W = \{a + bx + ce^x + dxe^x \mid a, b, c, d \in R\}$ of the space of continuous functions, and let $D_x = \frac{d}{dx}$ be the differential operator on W. Find the matrix for D_x relative to the basis B.

2. (20%) On $\mathbf{R}^3$ the map $h: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix}$. Find the representation $\mathbf{Rep}_{E_3E_3}(h)$ with respect to the standard basis $E_3 = \{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}$ of $\mathbf{R}^3$. Find also the representation $\mathbf{Rep}_{BB}(h)$ with respect to another basis $B = \{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$.

3. (15%) Use the Gram-Schmidt orthonormalization process to find an orthogonal matrix P such that $P^T AP$ diagonalizes the symmetric matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

4. (15%) For the vector field $\vec{v} = A\vec{r}$, that is pointing away from the origin with a magnitude proportional to the distance from the origin, express this in rectangular components and compute its divergence.

5. (15%) Evaluate the surface integral $\oint \vec{v} \cdot d\vec{A}$ of $\vec{v} = \hat{r}Ar^2 \sin^2\theta + \hat{\theta}Br\cos\theta\sin\phi$ over the surface of the sphere centered at the origin and of radius R.

6. (20%) The vector potential is not unique, as you can add an arbitrary gradient to it without affecting its curl. Suppose that $\vec{B} = \nabla \times \vec{A}$ with $\vec{A} = \hat{x}\alpha xyz + \hat{y}\beta x^2 z + \hat{z}\gamma xyz^2$.

   Find a function $f(x, y, z)$ such that $\vec{A}' = \vec{A} + \nabla f$ has the z-component identically zero. Do you get the same $\vec{B}$ by taking the curl of $\vec{A}$ and of $\vec{A}'$?